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On the local convergence of a family of two-step iterative methods for solving nonlinear equations

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A local convergence analysis for a generalized family of two step Secant-like methods with frozen operator for solving nonlinear equations is presented. Unifying earlier methods such as Secant's, Newton, Chebyshev-like, Steffensen and other new variants the family of iterative schemes is built up, where a profound and clear study of the computational efficiency is also carried out. Numerical examples and an application using multiple precision and a stopping criterion are implemented without using any known root. Finally, a study comparing the order, efficiency and elapsed time of the methods suggested supports the theoretical results claimed.

1. Introduction

There are a great variety of iterative methods for solving a system of nonlinear equations $F(x) = 0$, where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$, and D is a non-empty open convex subset of $\mathbb{R}^m$ that contains a simple root α of F.

A classical iterative process for solving nonlinear equations is Chebyshev's method (see [1–3])

$$\begin{cases} x_0 \in D, \\ y_n = x_n - F'(x_n)^{-1} F(x_n) \\ x_{n+1} = y_n - \dfrac{1}{2} F'(x_n)^{-1} F''(x_n)\,(y_n - x_n)^2, \quad n \geq 0. \end{cases}$$

The above one-point iterative scheme depends explicitly on the two first derivatives of F. In [1], Ezquerro and Hernández present some modifications in Chebyshev's method by reducing in one the number of evaluations of the first derivative and maintaining third-order of convergence. It has the following form:

$$\begin{cases} x_0 \in D, \\ z_n = x_n - a\,F'(x_n)^{-1} F(x_n), \\ x_{n+1} = x_n - \dfrac{1}{a^2} F'(x_n)^{-1} \left((a^2 + a - 1)\,F(x_n) + F(z_n)\right), \quad n \geq 0. \end{cases}$$

Using the well-known Secant method [4], in [5] a generalization of it employing the divided difference operator of order one (namely, $B_n = [x_{n-1}, x_n; F]$) that substitutes the derivative of F ($F'(x_n) \equiv [x_n, x_n; F]$) is given. The authors call this family

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the Chebyshev–Secant-type method and it is defined by

$$\begin{cases} x_{-1},\ x_0 \in D, \\ z_n = x_n - a\, B_n^{-1} F(x_n), \\ x_{n+1} = x_n - B_n^{-1}\ \left(b\, F(x_n) + c\, F(z_n)\right), \quad n \geq 0, \end{cases}$$

where $a,\ b,\ c$ are non-negative parameters to be chosen so that the sequence $\{x_n\}$ converges to α with maximum local order of convergence.

The work presented in [6] analyzes free-derivative iterative processes considering the operator $C_n = [x_n, L(x_n); F]$ and they are called the Steffensen-type method:

$$\begin{cases} x_0 \in D, \\ z_n = x_n - a\, C_n^{-1} F(x_n), \\ x_{n+1} = x_n - C_n^{-1}\ \left(b\, F(x_n) + c\, F(z_n)\right), \quad n \geq 0, \end{cases}$$

where $L : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$.

Our goal in the present paper is to unify these methods by considering the following iterative family

$$\begin{cases} x_{-1}, \quad x_0 \in D, \\ z_n = x_n - a\, \Theta_n^{-1} F(x_n), \\ x_{n+1} = x_n - \Theta_n^{-1}\ \left(b\, F(x_n) + c\, F(z_n)\right), \quad n \geq 0, \end{cases} \tag{1}$$

where $\Theta_n = [G(x_{n-1}, x_n),\ H(x_{n-1}, x_n); F]$ and $G,\ H : D \subseteq \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. As can be observed, the above family includes as particular cases the preceding works [5,6] and other well-known algorithms. Furthermore, (1) also offers us the possibility to suggest new methods as we will see later on.

2. Local order of convergence

To obtain explicitly an expression of the inverse operator of Θ_n, we denote by J indistinctly both G or H. On account that a necessary condition about functions G and H is: $G(\alpha,\ \alpha) = H(\alpha,\ \alpha) = \alpha$, then from Taylor's formulae we get the error expression

$$\varepsilon_J = J(x_{n-1}, x_n) - \alpha = J_1\, e_{n-1} + J_2\, e_n + 1/2\left(J_{11}\, e_{n-1}^2 + J_{22}\, e_n^2 + 2J_{12}\, e_{n-1}\, e_n\right) + \cdots,$$

where $J_1 = \frac{\partial J}{\partial x_{n-1}}(\alpha,\ \alpha),\ J_2 = \frac{\partial J}{\partial x_n}(\alpha,\ \alpha),\ J_{11} = \frac{\partial^2 J}{\partial x_{n-1}^2}(\alpha,\ \alpha)$, etc.

The expression of Θ_n is given in the previous works appeared in [7,8]. Namely,

$$\Theta_n = [G(x_{n-1}, x_n),\ H(x_{n-1}, x_n); F] = \Gamma\left(I + A_2\,(\varepsilon_G + \varepsilon_H) + A_3\,(\varepsilon_G^2 + \varepsilon_G\varepsilon_H + \varepsilon_H^2) + A_4\,(\varepsilon_G^3 + \varepsilon_G^2\varepsilon_H + \varepsilon_G\varepsilon_H^2 + \varepsilon_H^3) + \cdots\right),$$

where $\Gamma = F'(\alpha) \in \mathfrak{L}(\mathbb{R}^m)$, $A_k = \frac{1}{k!}\,\Gamma^{-1} F^{(k)}(\alpha)$, $A_k \in \mathfrak{L}_k(\mathbb{R}^m, \mathbb{R}^m)$ and A_k is k-symmetric. From the preceding follows

$$\Theta_n^{-1} = \Big(I - A_2\,(\varepsilon_G + \varepsilon_H) - (A_3 - A_2^2)\,(\varepsilon_G^2 + \varepsilon_G\varepsilon_H + \varepsilon_H^2) + A_2^2\,\varepsilon_G\varepsilon_H + A_2\,(\varepsilon_G + \varepsilon_H)\cdot A_3\,(\varepsilon_G^2 + \varepsilon_G\varepsilon_H + \varepsilon_H^2) + A_3\,(\varepsilon_G^2 + \varepsilon_G\varepsilon_H + \varepsilon_H^2)\cdot A_2\,(\varepsilon_G + \varepsilon_H) - A_4\,(\varepsilon_G^3 + \varepsilon_G^2\varepsilon_H + \varepsilon_G\varepsilon_H^2 + \varepsilon_H^3)\Big)\,\Gamma^{-1},$$

where $A_2^2\,\varepsilon_G^2 = (A_2\,\varepsilon_G)^2$. Setting $E_n = z_n - \alpha$, $e_n = x_n - \alpha$ and using Taylor's development yield

$$F(x_n) = \Gamma\left(e_n + A_2\, e_n^2 + o(e_n^2)\right).$$

Subtracting α from both sides of the first equation of (1), we obtain

$$\begin{aligned} E_n &= e_n - a\,\Theta_n^{-1}\,\Gamma\left(e_n + A_2\, e_n^2 + o(e_n^2)\right) \\ &= (1-a)\, e_n + a\, A_2\,(G_1 + H_1)\, e_{n-1}\, e_n \\ &\quad + a\left(\frac{1}{2} A_2\,(G_{11} + H_{11}) + A_3\,(G_1^2 + H_1^2 + G_1 H_1) - (A_2\,(G_1 + H_1))^2\right) e_{n-1}^2\, e_n \\ &\quad + a\, A_2\,(G_2 + H_2 - I)\, e_n^2 + o(e_{n-1}^2\, e_n,\ e_n^2), \end{aligned} \tag{2}$$

where the following notation was used:

$$A_2\,(G_1 + H_1)\, e_{n-1}\, e_n = A_2\,(G_1 e_{n-1} + H_1 e_{n-1})\, e_n,$$
$$A_2\,(G_{11} + H_{11})\, e_{n-1}^2\, e_n = A_2\,(G_{11} e_{n-1}^2 + H_{11} e_{n-1}^2)\, e_n,$$
$$A_3\,(G_1^2 + H_1^2 + G_1 H_1)\, e_{n-1}^2\, e_n = A_3\big((G_1 e_{n-1})(G_1 e_{n-1}) + (H_1 e_{n-1})(H_1 e_{n-1}) - (G_1 e_{n-1})(H_1 e_{n-1})\big)\, e_n,$$