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Approximation of vector fields using discrete div–rot variational splines in a finite element space ☆☆

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ABSTRACT

This paper deals with an approximation problem concerning vector fields through the new notion of div–rot variational splines. The minimizing problem is addressed in a finite element space through the choice of some semi-norms based on decomposition of the divergence operator and vector fields into a form with a rotational part. We study the existence and the uniqueness of the solution of such a problem. Then, a convergence result and an estimation of the error are established. Some numerical and graphical examples are analyzed in order to prove the validity of our method. Furthermore, we compare and show how our method improves upon one existing in the literature.

1. Introduction

Vector field approximation is a problem arising in many scientific applications, in areas such as fluid mechanics, meteorology, optical flow analysis, electromagnetics and image processing; see for example [1] and some of its references.

In recent years, different techniques for the construction of a curve or surface have been developed—for example, interpolation or fitting with spline functions, based on the minimization of a certain functional in a Sobolev space in relation to certain Lagrange or Hermite data (see [2–6]). In [7] the authors studied the approximation of vector fields using thin splines with tension. In [8], they studied the splines under tension in a bounded domain, discussing error and convergence for interpolation using div–curl splines under tension of a vector field in the classical vectorial Sobolev space (see also [9]). In [10] the authors chose a semi-norm based on the decomposition of vector fields into rotational and gradient parts. They used the variational spline technique, determining the vectorial function which minimizes such semi-norms over all the functions in a suitable semi-Hilbert space interpolating the data. Meanwhile, the same authors in [11] define three-dimensional splines interpolating data while being irrotational or divergence free.

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In this context, we present an approximation problem involving the novel notion of div–rot variational splines. The minimizing problem is resolved by choosing some semi-norms based on the decomposition of a diverging operator and vector fields with a rotational part. We study the existence and the uniqueness of the solution of such a problem, and show how to compute it in practice. We then establish a convergence result to show that the theory of minimizing functional splines may be used for an approximation or for the fitting of a vector field controlled by divergence and rotation of the vector field. Such minimizing functionals combine semi-norms of Sobolev space with the expressions for divergence and rotation, which in turn are controlled by some parameters. Another advantage of this work is that it can be considered as a generalization of papers [2–6] to the study of approximating vector fields.

The remainder of this paper is organized as follows. In Section 2, we briefly recall some preliminary notation and results. Section 3 states the approximation problem and defines the notion of a discrete div–rot smoothing variational spline, while in Section 4 we show how to compute such a spline. A convergence theorem and an estimation of the error are studied in Section 5. In Section 6, some numerical and graphical examples are given, and a comparison with another method existing in the literature is established.

2. Notation and preliminaries

Let $n, m \in \mathbb{N}^*$, $\langle\ \cdot\ \rangle_n$ and $\langle\ \cdot\ ,\ \cdot\ \rangle_n$ denote, respectively, the Euclidean norm and the inner product in $\mathbb{R}^n$. Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $H^m(\Omega; \mathbb{R}^n)$ be the usual Sobolev space of (classes of) functions $\mathbf{u}$ belonging to $L^2(\Omega; \mathbb{R}^n)$, together with all their partial derivatives $\partial^{\boldsymbol{\alpha}}\mathbf{u}$, in the distribution sense, of order $|\boldsymbol{\alpha}| \le m$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $|\boldsymbol{\alpha}| = \alpha_1 + \dots + \alpha_n \le m$. This space is equipped with the inner semi-products

$$(\mathbf{u}, \mathbf{v})_{\ell,n} = \sum_{|\boldsymbol{\alpha}|=\ell} \int_\Omega \left\langle \partial^{\boldsymbol{\alpha}}\mathbf{u}(\mathbf{x}), \partial^{\boldsymbol{\alpha}}\mathbf{v}(\mathbf{x}) \right\rangle_n d\mathbf{x}, \quad 0 \le \ell \le m,$$

the semi-norms

$$|\mathbf{u}|_{\ell,n} = \left(\sum_{|\boldsymbol{\alpha}|=\ell} \int_\Omega \left\langle \partial^{\boldsymbol{\alpha}}\mathbf{u}(\mathbf{x}) \right\rangle_n^2 d\mathbf{x} \right)^{1/2}, \quad 0 \le \ell \le m,$$

and the corresponding norm

$$\|\mathbf{u}\|_{m,n} = \left(\sum_{|\boldsymbol{\alpha}|\le m} \int_\Omega \left\langle \partial^{\boldsymbol{\alpha}}\mathbf{u}(\mathbf{x}) \right\rangle_n^2 d\mathbf{x} \right)^{1/2}.$$

In $H^m(\Omega)$, which means $n = 1$, we simplify the previous notation to $(\mathbf{u}, \mathbf{u})_\ell$, $|\mathbf{u}|_\ell$ and $\|\mathbf{u}\|_m$, respectively.

We denote by $\mathcal{M}_{N,n}$ the space of real matrices with N rows and n columns equipped with the inner product

$$\langle T, B \rangle_{N,n} = \sum_{i,j=1}^{N,n} t_{ij} b_{ij},$$

and the corresponding norm

$$\langle T \rangle_{N,n} = \left(\langle T, T \rangle_{N,n} \right)^{1/2},$$

with $T = (t_{ij})_{\substack{1\le i\le N \\ 1\le j\le n}}$ and $B = (b_{ij})_{\substack{1\le i\le N \\ 1\le j\le n}}$ belonging to $\mathcal{M}_{N,n}$.

In the following, let Ω be a polyhedrical open subset of $\mathbb{R}^3$. We use the classical notation for divergence and rotational operators, given by

$$\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{3} \partial_i u_i$$

$$\operatorname{rot}\mathbf{u} = \nabla \times \mathbf{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$$

with $\mathbf{u} = (u_1, u_2, u_3)$, and $\times$ stands for the vector product in $\mathbb{R}^3$ and $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$, $i, j = 1, 2, 3$.

Moreover, we assume as given:

- a partition $\mathcal{T}$ of $\overline{\Omega}$ into tetrahedra or parallelepipeds K;
- a finite element space X constructed on $\mathcal{T}$ such that

$$X \text{ is a space of finite dimension } I \text{ of } H^m(\Omega) \cap C^k(\overline{\Omega}), \quad m \le k+1. \tag{1}$$

Let $V = X^3$ be the parametric finite element space obtained from X. From (1) we deduce that

$$V \subset H^m(\Omega; \mathbb{R}^3) \cap C^k(\overline{\Omega}; \mathbb{R}^3), \tag{2}$$

$C^k(\overline{\Omega}; \mathbb{R}^3)$ standing for the classical set of continuous functions of class C^k. For the construction of the parametric finite element space V and its properties, see [12].