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A hybrid finite difference/control volume method for the three dimensional poroelastic wave equations in the spherical coordinate system[☆,☆☆]

Wensheng Zhang[a,*], Li Tong[a], Eric T. Chung[b]

[a] LSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

[b] Department of Mathematics, The Chinese University of Hong Kong, Hong Kong

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A B S T R A C T

In this paper, we consider the numerical approximation of the three-dimensional poroelastic wave equations in the spherical coordinate system. One difficulty in the design of an efficient numerical scheme is that the problem is singular in the center and the polar axes of the computational domain. Nevertheless, we develop a hybrid finite difference/control volume method for solving this problem. Our method is explicit and is second order accurate in both space and time. Numerical results are shown to confirm the convergence rate of our method and the effectiveness to simulate wave propagation in poroelastic media in the spherical coordinate system.

1. Introduction

The simulation of elastic wave propagation in complex poroelastic media is an important research area due to its wide range of applications in various fields such as geophysics and petroleum engineering. For instance, in order to obtain useful insight for the exploration of hydrocarbon, the behavior of elastic waves propagating in fluid-saturated porous media is crucial. Biot's linear theory [1–4] has been used as a basis for solving wave propagation problem in fluid saturated porous media. This theory is established under the following assumptions: (1) the fluid phase is continuous so that disconnected pores are treated as if a single solid matrix; (2) the porous media is statistically isotropic, which means for all cross sections, the ratio of pore area to the solid occupied area is essentially constant; (3) the microscopic pore size is much smaller than the seismic wavelength; (4) the deformations are small, which guarantees linearity of the mechanical processes; (5) the solid matrix is elastic. By using a model based on the poroelastic wave equations, the effects of fluid, pressure, porosity and permeability between phases can be systematically taken into account and provides more accurate solutions that cannot be obtained through the use of Biot's linear theory, which consists of a pure elastic or acoustic wave equation.

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* Corresponding author. Tel.: +86 010 62627374.
E-mail addresses: zws@lsec.cc.ac.cn (W. Zhang), tongli@lsec.cc.ac.cn (L. Tong), tschung@math.cuhk.edu.hk (E.T. Chung).

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The finite difference method is perhaps the most popular and practical tool used in simulating acoustic and elastic wave propagation [5–11]. In literature, there are many numerical schemes for solving the Biot equations, for example, the classical finite difference schemes are developed in [12–14], the staggered-grid finite difference method in [15–18] and the discontinuous Galerkin method in [19]. All of these methods are of course based on rectangular computational domains. In some practical situations such as the wave scattering problem [20,21], the computational domain is a spherical domain or a semi-infinite half-space domain and computations in the spherical coordinate system are more suitable. Moreover, one needs to solve the poroelastic wave equations in unbounded domains, and thus one needs to impose some artificial boundary conditions. One option is the so-called exact nonreflecting boundary conditions [21–24]. The construction of such boundary conditions is based on a spherical computational domain and spherical harmonics. Hence, it is more convenient to solve the resulting problem in spherical coordinates. In this paper, we consider the numerical approximation of the three-dimensional poroelastic wave equations defined on a spherical computational domain in the spherical coordinate system. The finite difference method provides a fast and easy-to-implement numerical method for such a problem. The main difficulty in the design of an efficient numerical scheme is that the problem is singular in the center and the polar axes of the computational domain. It is the purpose of this paper to develop an efficient scheme for this problem. In particular, we will propose a hybrid finite difference/control volume method for computing a numerical solution of the problem. Our method is explicit and is second order accurate in both space and time, despite the singularities.

The paper is organized as follows. In Section 2, we will present the three dimensional poroelastic wave equations in the spherical coordinate system and the corresponding finite difference scheme in non-singular regions. Then in Section 3, we will give a control volume approach to tackle the singular region, and this is the backbone of our hybrid method. In Section 4, we present some numerical results to verify the effectiveness and the rate of convergence of our method. Finally, we give a conclusion.

2. Poroelastic wave equations in a spherical coordinate system

According to Biot's theory, wave propagation in a three dimensional statistically poroelastic medium is described by Biot equations [1–4]:

$$2\sum_j \frac{\partial}{\partial x_j}(\mu\sigma_{ij}) + \frac{\partial}{\partial x_i}(\lambda\sigma - \alpha M\xi) = \frac{\partial^2}{\partial t^2}(\rho u_i + \rho_f w_i), \tag{2.1}$$

$$\frac{\partial}{\partial x_i}(\alpha M e - M\xi) = \frac{\partial}{\partial t^2}(\rho_f u_i + \mathring{\rho} w_i) + \frac{\eta}{\kappa}\frac{\partial w_i}{\partial t}, \tag{2.2}$$

where $\mathring{\rho} = a\rho_f/\phi$ is the apparent density, and

$$M = \left(\frac{\phi}{K_f} + \frac{\alpha - \phi}{K_s}\right)^{-1}, \quad \alpha = 1 - \frac{K_b}{K_s}. \tag{2.3}$$

In Eqs. (2.1)–(2.3), the physical parameters of the medium are described as follows: μ is the shear modulus of the dry porous matrix; λ is the Lamé constant of the saturated matrix; ϕ is the porosity; κ is the permeability of the matrix; ρ is the overall density of the saturated medium given by $\rho = \phi\rho_f + (1-\phi)\rho_s$; ρ_f is the density of the pore fluid, ρ_s is the density of the solid grains; η is the viscosity of the pore fluid; a is the tortuosity of the matrix; K_s is the bulk modulus of the matrix material; K_f is the bulk modulus of the pore fluid; K_b is the bulk modulus of the dry porous frame.

The Eqs. (2.1) and (2.2) consist of six equations defined in a three dimensional medium for the six components u_i and w_i, $i = 1, 2, 3$, where u_i is the ith component of the displacement vector of the solid material, $w_i = \phi(U_i - u_i)$ is the ith component of the displacement vector of the pore fluid relative to that of the solid, and U_i is the displacement vector of the pore fluid. Moreover, $\sigma_{ij} = (\partial u_j/\partial x_i + \partial u_i/\partial x_j)/2$ is the strain tensor in the porous medium, $\xi = -\nabla\cdot\boldsymbol{w}$ is the dilatation of the relative motion between the fluid and the solid, and $e = \nabla\cdot\boldsymbol{u}$ is the dilatation of the solid motion.

Now we consider a spherical computational domain $\mathcal{B}$ centered at the origin with radius R and assume that the medium is homogeneous. We will need to represent a vector field in spherical coordinates. For a general vector field $\boldsymbol{f}(r,\vartheta,\phi,t)$, we use f^r, f^ϑ and f^ϕ to represent the components of $\boldsymbol{f}$ in the directions $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\vartheta}}$ and $\hat{\boldsymbol{\phi}}$ respectively, where $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\vartheta}}$ and $\hat{\boldsymbol{\phi}}$ are the unit vectors in the spherical coordinate system. Let $\boldsymbol{u} = (u^r, u^\vartheta, u^\phi, t)$ and $\boldsymbol{w} = (w^r, w^\vartheta, w^\phi, t)$. Then in the spherical coordinate system, Eqs. (2.1) and (2.2) can be written as:

$$\begin{aligned}(\mathring{\rho}\rho - \rho_f^2)\frac{\partial^2 u^r}{\partial t^2} &= (\mathring{\rho}\lambda + \mathring{\rho}\mu - \alpha\rho_f M)\frac{\partial}{\partial r}\Big(G(u^r, u^\theta, u^\phi)\Big)\\ &\quad + (\alpha\mathring{\rho}M - \rho_f M)\frac{\partial}{\partial r}\Big(G(w^r, w^\theta, w^\phi)\Big) + \mathring{\rho}\mu H(u^r) + \frac{\eta\rho_f}{\kappa}\frac{\partial w^r}{\partial t},\end{aligned} \tag{2.4}$$

$$\begin{aligned}(\mathring{\rho}\rho - \rho_f^2)\frac{\partial^2 u^\vartheta}{\partial t^2} &= \frac{(\mathring{\rho}\lambda + \mathring{\rho}\mu - \alpha\rho_f M)}{r}\frac{\partial}{\partial\vartheta}\Big(G(u^r, u^\vartheta, u^\phi)\Big)\\ &\quad + \frac{(\alpha\mathring{\rho}M - \rho_f M)}{r}\frac{\partial}{\partial\vartheta}\Big(G(w^r, w^\vartheta, w^\phi)\Big) + \mathring{\rho}\mu\Big(H(u^\vartheta)\Big) + \frac{\eta\rho_f}{\kappa}\frac{\partial w^\vartheta}{\partial t},\end{aligned} \tag{2.5}$$