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Error estimates of anti-Gaussian quadrature formulae[☆]

Miodrag M. Spalević[*]

Department of Mathematics, University of Beograd, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia

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This work is dedicated to the memory of Professor Franz Peherstorfer.

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ABSTRACT

Anti-Gauss quadrature formulae associated with four classical Chebyshev weight functions are considered. Complex-variable methods are used to obtain expansions of the error in anti-Gaussian quadrature formulae over the interval $[-1, 1]$. The kernel of the remainder term in anti-Gaussian quadrature formulae is analyzed. The location on the elliptic contours where the modulus of the kernel attains its maximum value is investigated. This leads to effective L^∞-error bounds of anti-Gauss quadratures. Moreover, the effective L^1-error estimates are also derived. The results obtained here are an analogue of some results of Gautschi and Varga (1983) [11], Gautschi et al. (1990) [9] and Hunter (1995) [10] concerning Gaussian quadratures.

1. Introduction

Let w be a given nonnegative and integrable weight function on the interval $[-1, 1]$. Let us denote by p_k the monic polynomial of degree k, which is orthogonal to $\mathbb{P}_{k-1}$ ($\mathbb{P}_k$ denotes the set of polynomials of degree at most k) with respect to w, i.e.

$$\int_{-1}^{1} x^j p_k(x) w(x) \mathrm{d}x = 0, \quad j = 0, 1, \ldots, k-1,$$

and let us recall that (p_k) satisfies a three-term recurrence relation of the form

$$p_{k+1}(x) = (x - a_k) p_k(x) - b_k p_{k-1}(x), \quad k = 0, 1, \ldots, \tag{1.1}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and the b_k's have the property of being positive.

The unique interpolatory quadrature formula with n nodes and the highest possible degree of exactness $2n - 1$ is the Gaussian formula with respect to the weight w,

$$\int_{-1}^{1} f(x) w(x) \mathrm{d}x = G_n[f] + E_n(f), \quad G_n[f] = \sum_{j=1}^{n} \lambda_j^G f\left(x_j^G\right) \ (n \in \mathbb{N}). \tag{1.2}$$

In [1], Laurie introduced quadrature rules that he referred to as anti-Gauss associated with the weight w,

$$\int_{-1}^{1} f(x) w(x) \mathrm{d}x = A_{n+1}[f] + R_{n+1}(f), \quad A_{n+1}[f] = \sum_{j=1}^{n+1} \lambda_j^A f\left(x_j^A\right) \ (n \in \mathbb{N}). \tag{1.3}$$

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* Tel.: +381 11 3302 334; fax: +381 11 3370 364.
E-mail address: mspalevic@mas.bg.ac.rs.

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This is an $(n+1)$-point interpolatory formula of degree $2n-1$ which integrates polynomials of degree up to $2n+1$ with an error equal in magnitude but opposite sign to that of the n-point Gaussian formula (1.2). Its intended application is to estimate the error incurred in Gaussian integration by halving the difference between the results obtained from the two formulae. Concerning with this and related problematic there appeared several papers in the last time, see [2–8]. Laurie [1] showed that an anti-Gaussian quadrature formula has positive weights and that its nodes are in the integration interval (except that for some weight functions, at most two of the nodes may be outside the integration interval) and are interlaced by those of the corresponding Gaussian formula. The anti-Gaussian formula is as easy to compute as the $(n+1)$-point Gaussian formula. Finally, the anti-Gaussian quadrature formula (1.3) is based on the zeros of polynomial

$$\pi_{n+1} = p_{n+1} - b_n p_{n-1}, \tag{1.4}$$

which is orthogonal subject to the linear functional $2\int[\cdot]w(x)\mathrm{d}x - G_n[\cdot]$.

In this paper w represents one of four classical Chebyshev weight functions:

$$w_1(t) = \frac{1}{\sqrt{1-t^2}}, \qquad w_2(t) = \sqrt{1-t^2}, \qquad w_3(t) = \sqrt{\frac{1+t}{1-t}}, \qquad w_4(t) = \sqrt{\frac{1-t}{1+t}}.$$

In these cases all nodes of the anti-Gauss quadrature formula (1.3), i.e., all zeros of the corresponding polynomial π_{n+1}, belong to the interval $[-1, 1]$. They are, in the same time, the Kronrod nodes (see [1]).

2. On the remainder term of anti-Gauss quadrature formulae for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let D be its interior. If integrand f is analytic on D and continuous on $\overline{D}$, and if all nodes of anti-Gauss quadrature formula belong to the interval $[-1, 1]$, then the remainder term $R_{n+1}(f)$ in (1.3) admits the contour integral representation

$$R_{n+1}(f) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} K_{n+1}(z) f(z)\mathrm{d}z. \tag{2.1}$$

The *kernel* is given by

$$K_{n+1}(z) = \frac{\varrho_{n+1}(z)}{\pi_{n+1}(z)}, \quad z \notin [-1, 1], \tag{2.2}$$

where

$$\varrho_{n+1}(z) = \int_{-1}^{1} \frac{\pi_{n+1}(x)}{z-x} w(x)\mathrm{d}x.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n+1}(\overline{z})| = |K_{n+}(z)|$. If the weight function w is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n+1}(-\overline{z})| = |K_{n+1}(z)|$ (see [9]).

In many papers error bounds of $|E_n(f)|$, i.e., of the modulus of the remainder term in Gauss quadrature formula (1.2), where f is an analytic function, are considered. Two choices of the contour Γ have been widely used:

- a circle C_r with a center at the origin and a radius r (>1), i.e., $C_r = \{z \mid |z| = r\}$, $r > 1$, and
- an ellipse $\mathcal{E}_\rho$ with foci at the points ∓ 1 and a sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}\left(\xi + \xi^{-1}\right),\ \xi = \rho\, \mathrm{e}^{\mathrm{i}\theta},\ 0 \le \theta \le 2\pi \right\}. \tag{2.3}$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1, 1]$, while with increasing ρ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1. In this paper we take Γ to be the ellipse $\mathcal{E}_\rho$.

The integral representation (2.1), for the remainder term in the anti-Gauss quadrature formula (1.3), leads to a general error estimate, by using Hölder's inequality,

$$\begin{aligned} |R_{n+1}(f)| &= \frac{1}{2\pi} \left| \oint_{\mathcal{E}_\rho} K_{n+1}(z) f(z)\mathrm{d}z \right| \\ &\le \frac{1}{2\pi} \left(\oint_{\mathcal{E}_\rho} |K_{n+1}(z)|^r |\mathrm{d}z| \right)^{1/r} \left(\oint_{\mathcal{E}_\rho} |f(z)|^{r'} |\mathrm{d}z| \right)^{1/r'}, \end{aligned}$$

i.e.,

$$|R_{n+1}(f)| \le \frac{1}{2\pi} \|K_{n+1}\|_r \|f\|_{r'},$$