Journal of Computational and Applied Mathematics 236 (2012) 3605–3612

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convergence of a finite difference method for solving 2D parabolic interface problems ☆

Dejan R. Bojović [a,*], Boško S. Jovanović [b]

[a] University of Kragujevac, Faculty of Science, R. Domanovića 12, 34000 Kragujevac, Serbia

[b] University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia

ARTICLE INFO

Article history:
Received 21 September 2010
Received in revised form 21 April 2011

MSC:
65M12
65M15

Keywords:
Interface problem
Convergence
Sobolev norm

ABSTRACT

The convergence of a difference scheme for solving two-dimensional parabolic interface problems with variable coefficients is investigated. An estimate of the rate of convergence in a special discrete $\widetilde{W}_2^{2,1}$ Sobolev norm, compatible with the smoothness of the coefficients and solution, is obtained.

1. Introduction

The finite difference method is one of the basic tools used in the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has weak global regularity and it is impossible to establish convergence of finite difference schemes using the classical Taylor series expansion. Often, the Bramble–Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [1–3].

Following Lazarov et al. [3], a convergence rate estimate of the form

$$\|u - v\|_{W_{2,h}^k} \le Ch^{s-k}\|u\|_{W_2^s}, \quad s > k,$$

is called **compatible** with the smoothness (regularity) of the solution u of the boundary value problem. Here v is the solution of the discrete problem, h is the spatial mesh step, W_2^s and $W_{2,h}^k$ are Sobolev spaces of functions with continuous and discrete arguments, respectively, and C is a constant which does not depend on u and h. For the parabolic case typical estimates are of the form

$$\|u - v\|_{W_{2,h\tau}^{k,k/2}} \le C(h + \sqrt{\tau})^{s-k}\|u\|_{W_2^{s,s/2}}, \quad s > k,$$

where τ is the time step. In the case of equations with variable coefficients the constant C in the error bounds depends on the norms of the coefficients (see, for example, [2,4,5]).

One interesting class of parabolic problems model processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow at the singular point is proportional to the time derivative of the temperature [6]. Such problems are nonstandard and the classical tools of the theory of finite difference schemes are difficult to apply in their convergence analysis.

☆ This work was supported in part by the Serbian Ministry of Science (Research Project: "Approximation of linear operators" (No. #144005)).

* Corresponding author.
E-mail addresses: bojovicd@ptt.rs (D.R. Bojović), bosko@matf.bg.ac.rs (B.S. Jovanović).

0377-0427/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2011.04.042

In the present paper a finite difference scheme approximating the two-dimensional initial–boundary value problem for the heat equation with concentrated capacity is derived. A special Sobolev norm (corresponding to the norm $W_2^{2,1}$ for a classical heat-conduction problem) is constructed. In this norm, a convergence rate estimate, compatible with the smoothness of the solution of the boundary value problem, is obtained.

Note that the convergence to classical solutions is studied in [7,8]. The one-dimensional parabolic problem with a weak solution is studied in [9–11]; the 2D parabolic problem is considered in [12].

2. Preliminary results

Let H be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and S an unbounded self-adjoint positive definite linear operator, with domain $D(S)$ dense in H. It is easy to see that the product $(u,v)_S=(Su,v)\,(u,v\in D(S))$ satisfies the axioms of the inner product. The closure of $D(S)$ in the norm $\|u\|_S=(u,u)_S^{1/2}$ is a Hilbert space $H_S\subset H$. The inner product (u,v) continuously extends to $H_S^*\times H_S$, where $H_S^*=H_{S^{-1}}$ is the dual space for H_S. The spaces H_S, H and $H_{S^{-1}}$ form a Gelfand triple $H_S\subset H\subset H_{S^{-1}}$, with continuous imbedding. The operator S extends to the map $S:H_S\to H_S^*$. There exists an unbounded self-adjoint positive definite linear operator $S^{1/2}$ such that $D(S^{1/2})=H_S$ and $(u,v)_S=(Su,v)=(S^{1/2}u,S^{1/2}v)$. We also define the Sobolev spaces $W_2^s(a,b;H)$, $W_2^0(a,b;H)=L_2(a,b;H)$ of the functions $u=u(t)$ mapping interval $(a,b)\subset R$ into H (see [13,14]).

Let A and B be unbounded self-adjoint positive definite linear operators, $A\neq A(t)$, $B\neq B(t)$, in the Hilbert space H, in general noncommutative, with $D(A)$ dense in H and $H_A\subset H_B$. We consider the following abstract Cauchy problem (cf. [15,14]):

$$B\frac{\mathrm{d}u}{\mathrm{d}t}+Au=f(t),\quad 0<t<T;\qquad u(0)=u_0,\tag{1}$$

where $f(t)$ and u_0 are given and $u(t)$ is an unknown function with values in H. The following proposition holds (see [9]).

Lemma 1. *The solution of the problem (1) satisfies the a priori estimate*

$$\int_0^T\left(\|Au(t)\|_{B^{-1}}^2+\left\|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right\|_B^2\right)\mathrm{d}t\le C\left(\|u_0\|_A^2+\int_0^T\|f(t)\|_{B^{-1}}^2\,\mathrm{d}t\right),\tag{2}$$

provided that $u_0\in H_A$ *and* $f\in L_2(0,T;H_{B^{-1}})$.

Analogous results hold for operator-difference schemes. Let H_h be a finite dimensional real Hilbert space with inner product $(\cdot,\cdot)_h$ and norm $\|\cdot\|_h$. Let $A_h\neq A_h(t)$ and $B_h\neq B_h(t)$ be self-adjoint positive linear operators defined on H_h, and in the general case noncommutative. By H_{S_h}, where $S_h=S_h^*>0$, we denote the space with inner product $(y,v)_{S_h}=(S_hy,v)_h$ and norm $\|y\|_{S_h}=(S_hy,y)_h^{1/2}$.

Let ω_τ be a uniform mesh on $(0,T)$ with the step size $\tau=T/m$, $\omega_\tau^-=\omega_\tau\cup\{0\}$, $\omega_\tau^+=\omega_\tau\cup\{T\}$ and $\overline{\omega}_\tau=\omega_\tau\cup\{0,T\}$. Further we shall use standard notation from the theory of the difference schemes [16,4]. In particular we set

$$v_{\bar t}=v_{\bar t}(t)=\frac{v(t)-v(t-\tau)}{\tau},\qquad v_t=v_t(t)=\frac{v(t+\tau)-v(t)}{\tau}=v_{\bar t}(t+\tau).$$

We will consider the simplest implicit operator-difference scheme

$$B_hv_{\bar t}+A_hv=\varphi(t),\quad t\in\omega_\tau^+;\qquad v(0)=v_0,\tag{3}$$

where v_0 is a given element of H_h, $\varphi(t)$ is known and $v(t)$ is an unknown mesh function with values in H_h. The following analog of Lemma 1 holds true (see [2,17]).

Lemma 2. *For the solution of the problem (3) the following estimate holds:*

$$\tau\sum_{t\in\overline{\omega}_\tau}{}'\|A_hv(t)\|_{B_h^{-1}}^2+\tau\sum_{t\in\omega_\tau^+}\|v_{\bar t}(t)\|_{B_h}^2\le C\left(\|v_0\|_{A_h}^2+\tau\|A_hv_0\|_{B_h^{-1}}^2+\tau\sum_{t\in\omega_\tau^+}\|\varphi(t)\|_{B_h^{-1}}^2\right)$$

where we define

$$\sum_{t\in\overline{\omega}_\tau}{}'w(t)=\frac{w(0)}{2}+\sum_{t\in\omega_\tau}w(t)+\frac{w(T)}{2}.$$

We also need the next result (see [18]):

Lemma 3. *For* $f\in W_p^1(0,1)$, $p>1$ *and* $\varepsilon\in(0,1)$ *the following estimate holds:*

$$\|f\|_{L_p(0,\varepsilon)}\le C\varepsilon^{1/p}\|f\|_{W_p^1(0,1)}.$$