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On local convergence of a symmetric semi-discrete scheme for an abstract analogue of the Kirchhoff equation

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ABSTRACT

The present work considers a nonlinear abstract hyperbolic equation with a self-adjoint positive definite operator, which represents a generalization of the Kirchhoff string equation. A symmetric three-layer semi-discrete scheme is constructed for an approximate solution of a Cauchy problem for this equation. Value of the gradient in the nonlinear term of the scheme is taken at the middle point. It makes possible to find an approximate solution at each time step by inverting the linear operator. Local convergence of the constructed scheme is proved. Numerical calculations for different model problems are carried out using this scheme.

0. Introduction

Existence and uniqueness issues for local as well as global solutions of initial-boundary problems for the Kirchhoff string equation were first studied by Bernstein in 1940 (see [1]). The issues of solvability of the classical and generalized Kirchhoff equations were later considered by many authors: Arosio, Panizzi [2], Arosio and Spagnolo [3], Berselli, Manfrin [4], D'Ancona, Spagnolo [5,6], Manfrin [7], Medeiros [8], Liu, Rincon [9], Matos [10] and Nishihara [11]. To approximate the solutions of initial-boundary value problems for classical and generalized Kirchhoff equations the following works are dedicated: Christie, Sanz-Serna [12], Peradze [13] and Rogava, Tsiklauri [14]. As far as we know, issues of the approximate solution to an abstract analogue of the Kirchhoff string equation are less studied.

We consider a nonlinear abstract hyperbolic equation with a self-adjoint positive definite operator which represents a generalization of the Kirchhoff string equation (it also comprises a spatial multi-dimensional case). We search the approximate solution to a Cauchy problem for this equation using a symmetric three-layer semi-discrete scheme. Value of the gradient in the nonlinear term of the equation is taken at the middle point. It makes possible to find an approximate solution at each time step by inverting the linear operator.

Investigation of convergence of the constructed semi-discrete scheme for the abstract Kirchhoff string equation with an operator A is based on two facts: (a) $(u_k - u_{k-1})/\tau$ and $A^{1/2}u_k$ are uniformly bounded (u_k is an approximate solution); (b) $A^{1/2}(u_k - u_{k-1})/\tau$ and Au_k are locally bounded. Fact (b) is proved using a nonlinear inequality. Facts (a) and (b) allow to use Gronwall's lemma and prove a local convergence of the approximate solution. Convergence rate of the considered scheme is equal to two.

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We will make a remark about fact (b). The difficulties, which accompany the proof of existence and uniqueness theorems for the solution of the Cauchy–Dirichlet problem for the Kirchoff equation, are given clearly in the work of Arosio and Panizzi (see [2]). The problem that relates with obtaining a priori estimate in Sobolev spaces of low order and with a definition of time interval, in our opinion, is so said "genetic" and it is inherited by the discrete problem, that also takes place in our case.

The results of the numerical calculations of test problems are presented at the end of the work. Note that with regard to the spatial coordinate there is used a three-point difference scheme with the fourth order accuracy, that allows to reduce significantly a number of divisions with regard to spatial coordinate. On basis of numerical experiments, convergence rate of the scheme is practically stated and it is shown that the constructed scheme describes well the behavior of oscillating solution.

1. Statement of the problem and semi-discrete scheme

Let us consider the Cauchy problem for an abstract hyperbolic equation in the Hilbert space H:

$$\frac{d^2u(t)}{dt^2} + a\left(\left\|A^{1/2}u\right\|^2\right)Au(t) = f(t), \quad t \in [0, T], \tag{1.1}$$

$$u(0) = \varphi_0, \qquad \frac{du(0)}{dt} = \varphi_1. \tag{1.2}$$

where A is a self-adjoint (A does not depend on t), positive definite (generally unbounded) operator with the definition domain $D(A)$, which is everywhere dense in H, i.e. $\overline{D(A)} = H,\ A = A^*$ and

$$(Au, u) \geq \nu \|u\|^2, \quad \forall u \in D(A),\ \nu = \text{const} > 0.$$

Here $\|\cdot\|$ and $(\cdot, \cdot)$ denote the norm and scalar product in H, respectively; $a\left(\left\|A^{1/2}u\right\|^2\right) = \lambda + \left\|A^{1/2}u\right\|^2, \lambda > 0$; φ_0 and φ_1 are the given vectors from H; $u(t)$ is a continuous, twice continuously differentiable, sought function with values in H and $f(t)$ is the given continuous function with values in H.

Analogously to the linear case (see [15, p. 301]), the vector function $u(t)$ with values in H, defined on the interval $[0, T]$ is called a solution of the problem (1.1)–(1.2) if it satisfies the following conditions: (a) $u(t)$ is twice continuously differentiable in the interval $[0, T]$; (b) $u(t) \in D\left(A^2\right)$ for any t from $[0, T]$ and the function $A^2u(t)$ is continuous; (c) $u(t)$ satisfies the Eq. (1.1) on the interval $[0, T]$ and the initial condition (1.2). Here continuity and differentiability is meant by metric H.

Eq. (1.1) is an abstract analogue of the nonlinear Kirchhoff string equation:

$$\frac{\partial^2 u}{\partial t^2} = \left(\lambda + \int_0^L u_\xi^2(\xi, t)\, d\xi\right)\frac{\partial^2 u}{\partial x^2} + f(t).$$

We search for a solution to the problem (1.1)–(1.2) by the following semi-discrete scheme:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + a\left(\left\|A^{1/2}u_k\right\|^2\right)\frac{Au_{k+1} + Au_{k-1}}{2} = f_k, \tag{1.3}$$

where $k = 1, \ldots, n-1, \tau = T/n\ (n > 1), f_k = f(t_k), t_k = k\tau, u_0 = \varphi_0$.

As an approximate solution $u(t)$ of problem (1.1)–(1.2) at point t_k we declare u_k, $u(t_k) \approx u_k$.

In order to carry out a calculation using the scheme (1.3) it is necessary to know the starting vectors u_0 and u_1. The vector u_0 is given ($u_0 = \varphi_0$) and u_1 should be defined approximately. As is well known, to define the vector u_1 it is necessary to expand the exact solution $u(t)$ in Taylor series at the point $t = \tau$ and keep at least the first two terms, i.e., obtain: $u_1 = \varphi_0 + \tau\varphi_1$. To achieve higher order accuracy, we need to keep the first three terms (including the second order derivative) in the expansion of $u(\tau)$. The second order derivative of the function $u(t)$ at the point $t = 0$ can be defined from the Eq. (1.1) taking into account the initial conditions (1.2). Finally we obtain:

$$u_1 = \varphi_0 + \tau\varphi_1 + \frac{\tau^2}{2}\varphi_2, \quad \varphi_2 = f_0 - a\left(\left\|A^{1/2}\varphi_0\right\|^2\right)A\varphi_0. \tag{1.4}$$

If we insert the values of u_0 and u_1 in the Eq. (1.3), we obtain the following linear equation defining the vector u_2:

$$\left(I + \tau^2 a_1 A\right)u_2 = g, \tag{1.5}$$

where scalar $a_1 = \frac{1}{2}a\left(\left\|A^{1/2}u_1\right\|^2\right)$ and the right-hand side g are known.

Since, according to the condition, A is a self-adjoint positive definite operator and $a_1 > 0$, therefore the operator $I + \tau^2 a_1 A$ will be also a positive definite and self-adjoint. From here it follows that the operator $I + \tau^2 a_1 A$ is continuously invertible, i.e. the Eq. (1.5) has a unique solution for each g from H ($u_2 = \left(I + \tau^2 a_1 A\right)^{-1} g$), which depends continuously on the right-hand side. Analogously to u_2 we find u_k ($k > 2$), using u_{k-1} and u_{k-2}. Thus, realization of the scheme (1.3) is reduced to solving of a linear problem on each time layer.