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On some expansions for the Euler Gamma function and the Riemann Zeta function

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ABSTRACT

In the present paper we introduce some expansions which use the falling factorials for the Euler Gamma function and the Riemann Zeta function. In the proofs we use the Faá di Bruno formula, Bell polynomials, potential polynomials, Mittag-Leffler polynomials, derivative polynomials and special numbers (Eulerian numbers and Stirling numbers of both kinds). We investigate the rate of convergence of the series and give some numerical examples.

1. Introduction

Let us first recall some basic facts concerning special numbers and expansions. By $\left[{n \atop k} \right]$ we denote the Stirling number of the first kind (the number of ways of partitioning a set of n elements into k nonempty cycles, see [1]). It is set $\left[{n \atop 0} \right] = 0$ if $n > 0$, $\left[{0 \atop 0} \right] = 1$, $\left[{n \atop k} \right] = 0$ for $k > n$ or $k < 0$. The Stirling numbers of the first kind fulfill the recurrence formula

$$\left[{n \atop k} \right] = (n-1) \left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right]. \tag{1}$$

If $n > k$ then using formula (1) in each step to the last term of the resulting sum we get

$$\left[{n \atop k} \right] = \sum_{j=1}^{k} (n-j) \left[{n-j \atop k+1-j} \right]. \tag{2}$$

Stirling numbers of the first kind have the following generating function (see [2, pp. 50 and 135])

$$(1-t)^{-u} = 1 + \sum_{1 \le k \le n} \left[{n \atop k} \right] \frac{t^n}{n!} u^k. \tag{3}$$

We use common notations for the falling factorial

$$(x)_k = x(x-1) \cdots (x-k+1)$$

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and for the rising factorial (Pochhammer's symbol)

$$x^{(k)} = x(x+1)\cdots(x+k-1).$$

We denote by $B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ (see [3], [2, p. 133]) the exponential partial Bell polynomials in infinite number of variables $x_1, x_2, x_3, \ldots$. The polynomials are defined by the formal double series expansion in variables t and u

$$\exp\left(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n\geq 1} \frac{t^n}{n!}\left\{\sum_{k=1}^{n} u^k B_{n,k}(x_1, x_2, \ldots)\right\}. \tag{4}$$

We denote by P_n^r the potential polynomials (see [2, Theorem B p. 141]) which are defined for each complex number r by

$$\left(1 + \sum_{n\geq 1} g_n \frac{t^n}{n!}\right)^r = 1 + \sum_{n\geq 1} P_n^r \frac{t^n}{n!} \tag{5}$$

and

$$P_n^r = P_n^r(g_1, g_2, \ldots, g_n) = \sum_{1\leq k\leq n} (r)_k B_{n,k}(g_1, g_2, \ldots). \tag{6}$$

Formula (5) is a particular case of the Faá di Bruno formula and P_n^r (given by (6)) is the nth derivative (in a point $x = a$) of the function $(G(x))^r$, where $G(x)$ is given as the convergent power series $G(x) = 1 + \sum_{n\geq 1} g_n t^n/n!, \quad t = x - a,\ G(a) = 1$.

We investigate expansions, which involve the falling factorials, for the Euler Gamma function and for the integral (18). The last is expressed in terms of the Riemann Zeta function. For the coefficients of our series we give simple recurrence formulae. Some series for the Riemann Zeta function based on falling factorials have been studied, for example, in [4], who demonstrated the importance of such expansions. The coefficients of their expansion are expressed in terms of the values of the Zeta function in integers.

The article is organized as follows. In Section 2 we present the construction of the expansion, based on falling factorials, for the Euler Gamma function. For its coefficients we give the recurrence formula and an explicit formula involving the Stirling numbers of the first kind. In Section 3 we present the basic properties of the derivative polynomials which are used in the next sections. Section 4 is devoted to the construction of the expansion for integral (18). For the coefficients of the expansion we give the recurrence formula and an explicit formula, which uses the coefficients of the Mittag-Leffler polynomials. In Section 5 we examine the rate of convergence of the series introduced in Sections 2 and 4. Moreover, we show the results of two numerical experiments. The paper is concluded in Section 6.

2. The Euler Gamma function

Substituting in the integral

$$\Gamma(s+1) = \int_0^\infty x^s e^{-x} \mathrm{d}x$$

$x = -\log(1-t)$ we have

$$\Gamma(s+1) = \int_0^1 (-\log(1-t))^s \mathrm{d}t = \int_0^1 t^s \left(\frac{1}{t}\log\frac{1}{1-t}\right)^s \mathrm{d}t = \int_0^1 t^s \left(1 + \frac{t}{2} + \cdots\right)^s \mathrm{d}t. \tag{7}$$

Our first aim is to find the values of the Bell polynomials $B_{n,k}$ for the sequence $(1/2, 2!/3, 3!/4, \ldots)$. Using expansion (4) we get

$$\begin{aligned}
\exp\left\{u\left(\frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{4}t^3 + \cdots\right)\right\} &= e^{-u}\exp\left\{u\left(1 + \frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{4}t^3 + \cdots\right)\right\} \\
&= e^{-u}\exp\left\{\frac{u}{t}(-\log(1-t))\right\} = e^{-u}(1-t)^{-\frac{u}{t}} \\
&= e^{-u}\left\{1 + \sum_{1\leq k\leq n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^n}{n!}\left(\frac{u}{t}\right)^k\right\} = e^{-u}\left\{1 + \sum_{1\leq k\leq n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^{n-k}u^k}{n!}\right\} \\
&= \left(1 - \frac{u}{1!} + \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} - \cdots\right) + \sum_{j=0}^{\infty} \sum_{1\leq k\leq n} (-1)^j \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^{n-k}u^{k+j}}{n!j!}.
\end{aligned}$$

Putting $n - k = \alpha \geq 0, k + j = \beta \geq 1, (n = k + \alpha,\ j = \beta - k)$ we see that the coefficient of $t^\alpha u^\beta$ is

$$\sum_{k=1}^{\beta} \begin{bmatrix} k+\alpha \\ k \end{bmatrix} \frac{(-1)^{\beta-k}}{(k+\alpha)!(\beta-k)!} = \frac{1}{(\alpha+\beta)!}\sum_{k=1}^{\beta} (-1)^{\beta-k} \begin{bmatrix} k+\alpha \\ k \end{bmatrix} \binom{\alpha+\beta}{\beta-k}. \tag{8}$$