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Step size strategies for the numerical integration of systems of differential equations[☆]

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ABSTRACT

In this study, the step size strategies are obtained such that the local error is smaller than the desired error level in the numerical integration of a type of nonlinear equation system in interval $[t_0, T]$. The algorithms are given for calculating step sizes and numerical solutions according to these strategies. The algorithms are supported by the numerical examples.

1. Introduction

Selection of the step size is one of the most important concepts in numerical integration of differential equation systems. It is not practical to use constant step size in numerical integration. If the selected step size is large in numerical integration, the computed solution can diverge from the exact solution. And if the chosen step size is small, the calculation time, number of arithmetic operations, and the calculation errors start to increase. So, if the solution is changing rapidly, the step size should be chosen small. Inversely, if the solution is changing slowly, we should choose bigger step size.

The existence and uniqueness of the solution of the problem must be considered in the step size selection. Picard theorem and Lipschitz condition are well-known concepts which are related to the existence and uniqueness of the solution. Picard theorem can be found in [1–3] and Lipschitz condition can be situated in [1–6].

If the existence and uniqueness of the solution of the problem are known:

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{1.1}$$

the step size has been obtained as follows for Euler's method in [7]

$$h_i \le \left(\frac{2\delta_L}{\max\limits_{\tau\in[t_{i-1},t_i)} |z''(\tau)|} \right)^{\frac{1}{2}}, \quad (i = 1, 2, \ldots, n)$$

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and for the second-order Runge–Kutta method as follows,

$$h_i \le \left(\frac{12\delta_L}{M_{t_i}}\right)^{\frac{1}{3}}, \quad (i = 1, 2, \ldots, n)$$

where $\max_{\tau\in[t_{i-1},t_i)} |(f_{tt} + 2f.f_{tx} + f_x.f_t + f.f_x^2 + f^2 f_{xx})(\tau)| \le M_{t_i}$ in [7,8] such as local error is smaller than required error level δ_L in each step of the integration.

If the existence of the solution of the Cauchy problem given by Eq. (1.1) on region $D = \{(t, x) : |t - t_0| \le a, |x - x_0| \le b\}$ is unknown; the step size has been given as

$$h_i = \min\{a, b_{0i-1}/M_i\},$$

where y_i is the numerical solution obtained in the ith step, $z(t)$ is the solution of the Cauchy problem $z' = f(t, z), z(t_{i-1}) = y_{i-1}$, b_{i-1} is the upper bound of $|z - y_{i-1}|$ error, $b_{0i-1} = \min\{b_{0i-2}, b_{i-1}\}$, $D_{i-1} = \{(t, z) : |t - t_{i-1}| \le a, |z - y_{i-1}| \le b_{0i-1}\}$ and M_i is the upper bound of $f(t, z)$ on region D_{i-1} [7,9].

It is known that the solution of the Cauchy problem for the linear system

$$X'(t) = AX(t), \qquad X(t_0) = X_0 \tag{1.2}$$

exists and unique on the region $D = \{(t, X) : |t - t_0| \le T, |x_j - x_{j0}| \le b_j\}$, where $A = (a_{ij}) \in \boldsymbol{R}^{N\times N}$, $X(t) = (x_j(t))$, $X_0 = (x_{j0})$; $x_{j0} = x_j(t_0)$, $X(t), X_0$ and $b = (b_j) \in \boldsymbol{R}^N$. The step size has been obtained as

$$h_i \le \frac{1}{\alpha\sqrt[4]{N^5}}\left(\frac{2\delta_L}{\beta_{i-1}}\right)^{\frac{1}{2}} \tag{1.3}$$

such as local error is smaller than δ_L-error level for the Cauchy problem (1.2). Here, $\alpha = \max_{1\le i,j\le N} |a_{ij}|$, $\max_{1\le j\le N}(\sup_{t_{i-1}\le\tau_i<t_i} |z_j(\tau_i)|) \le \beta_{i-1}$ [10,11].

In addition, if it is desired that the local error is closer to δ_L in ith step of the numerical integration, the step size has been given as following

$$h_i = \gamma^{p-2}\hat{h}_i; \quad \hat{h}_i \le \frac{1}{\alpha\sqrt[4]{N^5}}\left(\frac{2\delta_L}{\beta_{i-1}}\right)^{\frac{1}{2}},$$

where $\hat{h}_i$ is the proposed step size in inequality (1.3), $\gamma > 1$ is a real number, δ_L is the required error level, $\alpha = \max_{1\le i,j\le N} |a_{ij}|$, $\max_{1\le j\le N}(\sup_{t_{i-1}\le\tau_{i,j}<t_i} |z_j(\tau_{i,j})|) \le \beta_{i-1}$ [10,11].

In this study, we aimed to develop step size strategies for the following system

$$X'(t) = AX(t) + \varphi(t, X) \tag{1.4}$$

using strategies mentioned above.

In Section 2; the concept of local error given in [7,8,12] as being defined for systems of differential equations and local error analysis has been examined. The step size strategy for linear systems has been remained. In Section 3, the step size strategies have been given for a type of nonlinear equation system. In Section 4; algorithms which calculate step sizes based on the given strategies and numerical solutions have been given. Finally, the effectiveness of the proposed method has been demonstrated by application to specific problems as the mechanical oscillator equation, the logistic growth model and the van der Pol equation in Section 5.

2. Preliminaries

Consider the Cauchy problem

$$X' = F(t, X), \qquad X(t_0) = X_0 \tag{2.1}$$

on the region $D = \{(t, X) : |t - t_0| \le T, |x_j - x_{j0}| \le b_j\}$, where $X(t) = (x_j(t))$, $X_0 = (x_{j0})$; $x_{j0} = x_j(t_0)$, $F(t, X) = (f_j)$; $f_j = f_j(t, x_1, x_2, \ldots, x_N)$, $F(t, X) \in C^1([t_0 - T, t_0 + T] \times \boldsymbol{R}^N)$, $X(t), X_0$ and $b = (b_j) \in \boldsymbol{R}^N$. We let give some basic concepts for the Cauchy problem given by the Eq. (2.1).

2.1. Basic concepts for system of differential equations

In this study, as a norm in $\boldsymbol{R}^N$ we use Euclidean norm, which is defined as follows

$$\|y\| = \sqrt{\sum_{j=1}^{N} y_j^2}, \quad y = (y_j) \in \boldsymbol{R}^N.$$

For every $A = (a_{ij}) \in \boldsymbol{R}^{N\times N}$, we use the Frobenius norm, i.e.

$$\|A\| = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2}.$$