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The approximation of functions from $L \log L(\log \log L)(S^N)$ by Fourier–Laplace series[☆]

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ABSTRACT

In this paper, a study of the approximation of functions from $L \log L(\log \log L)(S^N)$ by Fourier–Laplace series is performed. It is proved that the maximal operator of the Riesz means of the Fourier–Laplace series is bounded, from L_1 to $L \log L(\log \log L)(S^N)$. The result provides a natural and intrinsic characterization of the approximation of the functions by Fourier–Laplace series.

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1. Preliminaries and formulation of the main results

Let S^N be the unit sphere in R^{N+1}:

$$S^N = \{x \in R^{N+1} : |x|^2 = x_1^2 + x_2^2 + \cdots + x_{N+1}^2 = 1\}.$$

The sphere S^N is naturally equipped with a positive measure $d\sigma(x)$ and with an elliptic second-order differential operator Δ_s, namely the Laplace–Beltrami operator on the sphere. This operator is symmetric and nonnegative, and it can be extended to a nonnegative self-adjoint operator on the space $L_2(S^N)$, where $L_p(S^N)$ denotes the L_p-space associated with the measure $d\sigma(x)$ on the sphere. For the self-adjoint extension of the Laplace–Beltrami operator we use again the same symbol Δ_s, and by $\{\lambda_k\}$, $k = 0, 1, 2, \ldots$, we denote the sequence of the eigenvalues of the Laplace–Beltrami operator Δ_s, which is an increasing sequence of nonnegative eigenvalues $\lambda_k = k(k + N - 1)$, $k = 0, 1, 2 \ldots$, with finite multiplicities $a_0 = 1$, $a_1 = N$, $a_k = \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!} \approx k^{N-1}$, $k \geq 2$ (and written as such), tending to infinity. We denote by $Y_j^k(x)$ the eigenfunctions of the Laplace–Beltrami operator corresponding to λ_k:

$$\Delta_s Y_j^{(k)} = \lambda_k Y_j^{(k)}, \quad j = 1, 2, \ldots, a_k;\ k = 0, 1, 2, \ldots.$$

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The system of eigenfunctions of the Laplace–Beltrami operator is an orthonormal basis in $L_2(S^N)$ (see [1]). To any measurable function f we assign its spectral expansion:

$$f(x) \sim \sum_{k=0}^{\infty} Y_k(f,x), \tag{1.1}$$

where

$$Y_k(f,x) = \sum_{j=o}^{a_k} Y_j^{(k)}(x) \int_{S^N} f(y) Y_j^{(k)}(y) \mathrm{d}\sigma(y), \quad k = 0, 1, 2, \ldots.$$

The main purpose of this paper is to approximate the function f by the partial sums of (1.1):

$$E_n f(x) = \sum_{k=0}^{n} Y_k(f,x). \tag{1.2}$$

The Riesz means of the spectral expansions (1.2) can be defined by

$$E_n^{\alpha} f(x) = \int_{S^N} f(y) \Theta^{\alpha}(x,y,n) \mathrm{d}\sigma \tag{1.3}$$

where

$$\Theta^{\alpha}(x,y,n) = \sum_{k=0}^{n} \left(1 - \frac{\lambda_k}{\lambda_n}\right)^{\alpha} \sum_{j=o}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y). \tag{1.4}$$

We recall the standard notation: $\log^+ x = \log x$, if $x \geq 1$; otherwise $\log^+ x = 0$. The class of measurable functions satisfies the condition

$$\int_{S^N} |f(x)| \left(\log^+ |f(x)| \log^+ \log^+ |f(x)|\right) \mathrm{d}\sigma(x) < \infty,$$

which we denote by $L \log L(\log \log L)(S^N)$. It is not hard to see that $L \log L(\log \log L)(S^N) \subset L_1(S^N)$. We use the notation $\mu(B)$ for the Lebesgue measure of the set $B \subset S^N$. There is a simple connection between μ and σ:

$$\mu(B) = \int_B \mathrm{d}\sigma(x).$$

The maximal operator of Riesz means E_{λ}^{s}, which can be defined by

$$E_*^{\alpha} f(x) = \sup_{n \geq 1} |E_n^{\alpha} f(x)|, \tag{1.5}$$

plays an important role in estimating the error of the approximation.

Let us now proceed to the formulation of the basic results of the paper.

Theorem 1.1. *Let $f \in L \log L(\log \log L)(S^N)$; then for maximal operator of Riesz means at the critical index $\nu = \frac{N-1}{2}$ of the Fourier–Laplace series we have*

$$\mu\{|E_*^{\nu} f| > \lambda\} \leq \frac{K_1}{\lambda} \mu\{S^N\} + K_2 \frac{|\log \lambda|}{\lambda} \int_{S^N} |f(x)|[1 + (\log^+ |f(x)|) \log^+ \log^+ |f(x)|) \mathrm{d}\sigma(x),$$

where K_1 and K_2 do not depend on f and λ.

The proof of Theorem 1.1 is based on the following:

Theorem 1.2. *The maximal operator $E_*^{\frac{N-1}{2}}$ is a sublinear operator mapping $L_p(S^N)$, $p > 1$ into weak $L_p(S^N)$ such that*

$$\mu\left\{x \in S^N : \left|E_*^{\frac{N-1}{2}} f(x)\right| > \lambda\right\} \leq \left(\frac{A}{p-1} \frac{\|f\|_{L_p(S^N)}}{\lambda}\right)^p \tag{1.6}$$

for every $f \in L_p(S^N)$, $1 < p \leq 2$, with the constant A independent of f and p.

To establish this result we will estimate the maximal operator first in L_1 and L_2, and subsequently apply the interpolation theorem. The result in L_1 is:

Theorem 1.3. *Let $\alpha > \frac{N-1}{2}$; then for all $f \in L_1(S^N)$ we have*

$$\mu\{x \in S^N : |E_*^{\alpha} f(x)| > \lambda\} \leq \frac{c_{\alpha}}{\alpha - \frac{N-1}{2}} \frac{\|f\|_{L_1(S^N)}}{\lambda},$$

where the constant c_{α} does not depend on f and c_{α} remains bounded as $\alpha \to \frac{N-1}{2}$.

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