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Biquadratic finite volume element methods based on optimal stress points for parabolic problems[☆]

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ABSTRACT

In this paper, the semi-discrete and full discrete biquadratic finite volume element schemes based on optimal stress points for a class of parabolic problems are presented. Optimal order error estimates in H^1 and L^2 norms are derived. In addition, the superconvergences of numerical gradients at optimal stress points are also discussed. A numerical experiment confirms some results of theoretical analysis.

1. Introduction

The finite volume element (FVE) method [1–4], also named the generalized difference method [5–7], covolume method [8,9] or box method [10], has been becoming increasingly important as a discretization tool in lots of practical computations. The FVE method possesses not only the simplicity of a finite difference method but also the accuracy of a finite element method. More importantly, the method preserves local conservation of certain physical quantities. Readers are referred to [11–14] and references cited therein for some recent developments. By the approximation theory, we know that the numerical derivatives limited by the degree k of the approximate polynomials can obtain only k-th order accuracy; in general this estimate cannot be improved even if the solution possesses a higher smoothness. But this fact does not exclude the possibility that the approximation of derivatives may be of higher order accuracy at some special points, called optimal stress points. The FVE method based on optimal stress points for solving partial differential equations has been studied [15–18].

Recently, many researchers have focused on the FVE method in parabolic equations. The linear FVE method has been studied extensively in [2,8,19–24]. However, there is not a lot of literature on the high order FVE method. As regards the error estimates of the FVE method for the second order parabolic problems, we can borrow the theories and techniques of finite element methods to get basically parallel results. But there are certain difficulties requiring special treatment,

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such as the asymmetry of $(\cdot, \Pi_h^* \cdot)$; a technique dealing with the asymmetry of $(\cdot, \Pi_h^* \cdot)$ is given in [25]. In [26,18] Wang presented some high order FVE schemes for one-dimensional elliptic and parabolic differential equations. In [27] Yang and Yuan developed a symmetric biquadratic FVE scheme for nonlinear convection–diffusion problems and obtained an optimal H^1 error estimate. The dual partition ratio of the method is 1:4:1, i.e., each edge of an element in the primal partition Ω_h is partitioned into three segments so that the ratio of these segments is 1:4:1, this dual partition is different from the usually used ones in [5], where the partition ratio is 1:2:1 instead of 1:4:1. Using four interpolation optimal stress points on every rectangle element to construct a dual partition related to the primal partition, the authors in [16,17] developed a new class of biquadratic FVE methods for Poisson equations, and obtained the following optimal order L^2 error estimate by taking the advantage of optimal stress points:

$$\|u - u_h\|_0 \le Ch^3 \|u\|_4.$$

However, many numerical experiments indicate that both convergence rates of the biquadratic FVE methods carried out from the former two dual partitions are only $O(h^2)$ in the L^2 norm for elliptic equations, which are not optimal. In this paper, we will apply the same dual partition in [16,17] to establish some new FVE schemes for second order parabolic problems. We prove that these schemes not only possess optimal error estimates in H^1 and L^2 norms but also obtain the superconvergences of numerical gradients at optimal stress points.

The remainder of this paper is organized as follows. In Section 2, we introduce some necessary notations, and formulate the semi-discrete and full discrete FVE schemes. In Section 3, some auxiliary lemmas in order to analyze these schemes are proved. The semi-discrete and Crank–Nicolson FVE schemes are analyzed in Sections 4 and 5, respectively, including optimal order error estimates in the H^1 norm, L^2 norm and the superconvergences of numerical gradients at optimal stress points. In Section 6, a numerical experiment on the performance of the three methods, based on the different dual partitions, confirms that the new method improves the convergence rate of the other two methods.

Throughout this paper, the letter C denotes a generic positive constant independent of the mesh parameter and the time step size, and can have different values in different places.

2. The finite volume element methods

We consider the following second order parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (a(\mathbf{x})\nabla u) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \end{cases} \tag{2.1}$$

where $\Omega = (x_L, x_R) \times (y_L, y_R)$, $\mathbf{x} = (x, y)$, $a(\mathbf{x})$ is a positive real-valued function and $f(\mathbf{x}, t)$: $\Omega \times [0, T] \to \mathbb{R}$. The initial function u_0 is assumed to be smooth enough to insure the problem (2.1) has a unique solution in some Sobolev space.

For simplicity, we denote $u_t = \frac{\partial u}{\partial t}$. The weak formulation associated with (2.1) is: Find $u = u(\cdot, t) \in H_0^1(\Omega)$ $(0 < t \le T)$ such that

$$\begin{cases} (u_t, v) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega),\ 0 < t \le T, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{cases} \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ and $a(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a bilinear form defined by

$$a(u, v) = \int_\Omega a\nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}y, \quad \forall u, v \in H_0^1(\Omega). \tag{2.3}$$

Definition 2.1 (*Cf. [5]*)**.** Point x_0 is called an optimal stress point if there exists a $q \in [1, \infty]$ such that

$$|\overline{\triangledown}(u - \Pi_h u)(x_0)| \le Ch^{k+1-\frac{N}{q}} \|u\|_{k+2,q,E}, \quad \forall u \in W^{k+2,q}(E), \tag{2.4}$$

where E denotes the union of all the elements containing x_0, $\overline{\triangledown} v(x_0)$ the arithmetic mean of the values $\triangledown v(x_0)$ at every element in E, N the dimension of the region, and C a constant independent of the grid Ω_h and the solution u.

In [28], we have clarified that the set of the interpolation optimal stress points for a one-dimensional Lagrange quadratic finite element is

$$N_2 = F\hat{N}_2,$$

where F is the invertible affine mapping from the reference element $\hat{K} = [-1, 1]$ to the finite element K, and $\hat{N}_2$ is the set of the interpolation optimal stress points on $[-1, 1]$:

$$\hat{N}_2 = \left\{ -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\}.$$