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A posteriori error estimations for mixed finite-element approximations to the Navier–Stokes equations

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ABSTRACT

A posteriori estimates for mixed finite element discretizations of the Navier–Stokes equations are derived. We show that the task of estimating the error in the evolutionary Navier–Stokes equations can be reduced to the estimation of the error in a steady Stokes problem. As a consequence, any available procedure to estimate the error in a Stokes problem can be used to estimate the error in the nonlinear evolutionary problem. A practical procedure to estimate the error based on the so-called postprocessed approximation is also considered. Both the semidiscrete (in space) and the fully discrete cases are analyzed. Some numerical experiments are provided.

1. Introduction

We consider the incompressible Navier–Stokes equations

$$\begin{aligned} &u_t - \Delta u + (u\cdot\nabla)u + \nabla p = f,\\ &\operatorname{div}(u) = 0, \end{aligned} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a smooth boundary, subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1), u is the velocity field, p the pressure, and f a given force field. For simplicity in the exposition we assume, as in [1–5], that the fluid density and viscosity have been normalized by an adequate change of scale in space and time.

Let u_h and p_h be the semidiscrete (in space) mixed finite element (MFE) approximations to the velocity u and pressure p, respectively, solution of (1) corresponding to a given initial condition

$$u(\cdot, 0) = u_0. \tag{2}$$

We study the a posteriori error estimation of these approximations in the L^2 and H^1 norm for the velocity and in the $L^2/\mathbb{R}$ norm for the pressure. To do this, for a given time $t^* > 0$, we consider the solution $(\tilde u, \tilde p)$ of the Stokes problem

$$\left.\begin{aligned} &-\Delta\tilde u + \nabla\tilde p = f - \frac{\mathrm{d}}{\mathrm{d}t}u_h(t^*) - (u_h(t^*)\cdot\nabla)u_h(t^*)\\ &\operatorname{div}(\tilde u) = 0 \end{aligned}\right\}\quad \text{in } \Omega, \tag{3}$$
$$\tilde u = 0, \quad \text{on } \partial\Omega.$$

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In this paper we prove that $\tilde{u}$ and $\tilde{p}$ are approximations to u and p whose errors decay by a factor of $h|\log(h)|$ faster than those of u_h and p_h (h being the mesh size). As a consequence, the quantities $\tilde{u} - u_h$ and $\tilde{p} - p_h$, are asymptotically exact indicators of the errors $u - u_h$ and $p - p_h$ in the Navier–Stokes problem (1)–(2).

Furthermore, the key observation in the present paper is that (u_h, p_h) is also the MFE approximation to the solution $(\tilde{u}, \tilde{p})$ of the Stokes problem (3). Consequently, any available procedure to a posteriori estimate the errors in a Stokes problem can be used to estimate the errors $\tilde{u} - u_h$ and $\tilde{p} - p_h$ which, as mentioned above, coincide asymptotically with the errors $u - u_h$ and $p - p_h$ in the evolutionary NS equations. Many references address the question of estimating the error in a Stokes problem, see for example [6–12] and the references therein. In this paper we prove that any efficient or asymptotically exact estimator of the error in the MFE approximation (u_h, p_h) to the solution of the *steady* Stokes problem (3) is also an efficient or asymptotically exact estimator, respectively, of the error in the MFE approximation (u_h, p_h) to the solution of the *evolutionary* Navier–Stokes equations (1)–(2).

The analysis of the errors $u-\tilde{u}$ and $p-\tilde{p}$ is new and appears in this paper for the first time, although it follows closely [13], where MFE approximations to the Stokes problem (3) (the so-called postprocessed approximations) are considered with the aim of getting improved approximations to the solution of (1)–(2) at any fixed time $t^* > 0$. In [13], most of the results concern only quadratic and cubic elements. For this reason, in the present paper, some new results concerning first order finite elements that had not appeared before have also been included.

In this paper we will refer to $(\tilde{u}, \tilde{p})$ as infinite-dimensional postprocessed approximations (ID-postprocessed approximations). Of course, they are not computable in practice and they are only considered for the analysis of a posteriori error estimators. We remark that the Stokes reconstruction of [5] is exactly the ID-postprocessing approximation $(\tilde{u}, \tilde{p})$ in the particular case of a linear model. We prefer the term ID-postprocessed approximation for historical reasons and consistency with our previous published papers. In [5], the Stokes reconstruction is used to a posteriori estimate the errors of spatially semidiscrete approximations to a linear time-dependent Stokes problem.

The postprocessed approximations to the Navier–Stokes equations were first developed for spectral methods in [14–17], and also developed for MFE methods for the Navier–Stokes equations in [18,19,13].

For the sake of completeness, in the present paper we also analyze the use of the (computable) postprocessed approximations of [13] for a posteriori error estimation. The use of this kind of postprocessing technique to get a posteriori error estimations has been previously studied in [20–23] for nonlinear parabolic equations excluding the Navier–Stokes equations. For the analysis in the present paper we do not assume that the solution u of (1)–(2) possesses more than second-order spatial derivatives bounded in $L^2(\Omega)^d$ up to initial time $t = 0$, since demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations [2,3].

In the second part of the paper we consider a posteriori error estimations for the fully discrete MFE approximations $U_h^n \approx u_h(t_n)$ and $P_h^n \approx p_h(t_n)$, $(t_n = t_{n-1} + \Delta t_{n-1}$ for $n = 1, 2, \ldots, N)$ obtained by integrating in time with either the backward Euler method or the two-step backward differentiation formula (BDF). For this purpose, we define a Stokes problem similar to (3) but with the right-hand-side depending now on the fully discrete MFE approximation U_h^n (problem (70)–(71) in Section 4 below). We will call infinite-dimensional time-discrete postprocessed approximation (IDTD-postprocessed approximation) to the solution $(\widetilde{U}^n, \widetilde{P}^n)$ of this new Stokes problem. As before, $(\widetilde{U}^n, \widetilde{P}^n)$ is not computable in practice and it is only considered for the analysis of a posteriori error estimation. Again, the analysis of the errors $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ is new and appears for the first time in this paper, although following closely the analysis of [24].

Observe that in the fully discrete case (which is the case in actual computations) the task of estimating the error $u(t_n) - U_h^n$ of the MFE approximation becomes more difficult due to the presence of time discretization errors $e_h^n = u_h(t_n) - U_h^n$, which are added to the spatial discretization errors $u(t_n) - u_h(t_n)$. However we show in Section 4 that if temporal and spatial errors are not very different in size, the quantity $\widetilde{U}^n - U_h^n$ correctly estimates the spatial error because the leading terms of the temporal errors in $\widetilde{U}^n$ and U_h^n cancel out when subtracting $\widetilde{U}^n - U_h^n$, leaving only the spatial component of the error. This is a very convenient property that allows to use independent procedures for the tasks of estimating the errors of the spatial and temporal discretizations. More precisely, we mean that we can choose the tolerance for the temporal error and the tolerance for the spatial error approximately of the same size, in order to control both temporal and spatial errors in an adaptive way. We remark that the temporal error can be routinely controlled by resorting to well-known ordinary differential equations techniques. We refer the reader to [22], where analogous results were obtained for fully discrete finite element approximations to evolutionary convection–reaction–diffusion equations using the backward Euler method, and where an adaptive algorithm is proposed. The performance of an adaptive algorithm in time and space for the Navier–Stokes equations will be the subject of future research.

As in the semidiscrete case, a key point in our results is again the fact that the fully discrete MFE approximation (U_h^n, P_h^n) to the Navier–Stokes problem (1)–(2) is also the MFE approximation to the solution $(\widetilde{U}^n, \widetilde{P}^n)$ of the Stokes problem (70)–(71). As a consequence, we can use again any available error estimator for the Stokes problem to estimate the spatial error of the fully discrete MFE approximations (U_h^n, P_h^n) to the Navier–Stokes problem (1)–(2).

Computable mixed finite element approximations to $(\widetilde{U}^n, \widetilde{P}^n)$, the so-called fully discrete postprocessed approximations, were studied and analyzed in [24] where we proved that the fully discrete postprocessed approximations maintain the increased spatial accuracy of the semidiscrete approximations. The analysis in the second part of the present paper borrows in part from [24]. We also include error bounds for the L^2 norm of the difference between the temporal errors of the Galerkin and postprocessed approximations to the pressure, that had not been proved before. Finally, we propose a computable error