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Mass and momentum conservation of the least-squares spectral collocation method for the Navier–Stokes equations

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ABSTRACT

From the literature, it is known that the Least-Squares Spectral Element Method (LSSEM) for the stationary Stokes equations performs poorly with respect to mass conservation but compensates this lack by a superior conservation of momentum. Furthermore, it is known that the Least-Squares Spectral Collocation Method (LSSCM) leads to superior conservation of mass and momentum for the stationary Stokes equations. In the present paper, we consider mass and momentum conservation of the LSSCM for time-dependent Stokes and Navier–Stokes equations. We observe that the LSSCM leads to improved conservation of mass (and momentum) for these problems. Furthermore, the LSSCM leads to the well-known time-dependent profiles for the velocity and the pressure profiles. To obtain these results, we use only a few elements, each with high polynomial degree, avoid normal equations for solving the overdetermined linear systems of equations and introduce the Clenshaw–Curtis quadrature rule for imposing the average pressure to be zero. Furthermore, we combined the transformation of Gordon and Hall (transfinite mapping) with the least-squares spectral collocation scheme to discretize the internal flow problems.

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1. Introduction

Spectral methods (see, e.g., Canuto et al. [1], Gottlieb and Orszag [2,3] or Deville et al. [4]) employ global polynomials for the numerical solution of differential equations.

Hence, they give very accurate approximations for smooth solutions with relatively few degrees of freedom. For sufficiently smooth data, exponential convergence can be achieved.

If one deals with problems with non-smooth solutions the usual (global) spectral approach yields very poor approximation results. To avoid these difficulties the original domain can be decomposed into several sub domains and least-squares techniques can be applied; see e.g. [5–20]. Least-squares techniques for such problems offer theoretical and numerical advantages over the classical Galerkin type methods which must fulfill the well-posedness (or stability) criterion, the so called LBB condition. The advantage of least-squares techniques is that they lead to positive definite algebraic systems which circumvent the LBB stability condition; see, e.g. [21–25]. One very special least-squares technique is the least-squares spectral element method; see, e.g. [26,27,17–19]. These least-squares spectral element methods (see, e.g. [28]) for the Stokes problem were first introduced by Proot and Gerritsma in [16,17]. Spectral least-squares for the Navier–Stokes equations were first presented by Pontaza and Reddy in [13–15], followed by Proot and Gerritsma in [19]. Heinrichs investigated least-squares spectral collocation schemes in [6–9] that lead to symmetric and positive

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definite algebraic systems which circumvent the LBB stability condition. Furthermore, Heinrichs and Kattelans presented in [9,11] least-squares spectral collocation schemes where they improved the condition numbers of the algebraic systems, considered different types of decompositions of the domain and different interface conditions between the elements for the Stokes and Navier–Stokes equations. In [12] they have shown that the Least-Squares Spectral Collocation Method (LSSCM) leads to improved conservation of mass and momentum for an internal flow problem for the stationary Stokes equations. Within LSSCM we use spectral elements and a collocation approach on each element. Thus, LSSCM is a subset of LSSEM.

Here, we consider internal flow problems to investigate mass and momentum conservation of the LSSCM for the time-dependent Stokes equations and for the Navier–Stokes equations. A typical example of such a flow problem is a small channel of width h in which a cylinder with diameter d moves along the centerline of the channel, see e.g. [29,20,12].

In [29] it has been shown for the stationary Stokes equations that the Least-Squares Finite Element Method (LSFEM) leads to an unsatisfactory velocity profile along the smallest cross-section between the channel wall and the cylinder. Using this calculated velocity profile to calculate the mass flow through the cross-section it has been observed that the calculated mass flow is significantly lower than the mass inflow into the channel.

The important questions is:

Why are least-squares methods more susceptible to loss of mass conservation than, e.g., Galerkin-type methods?

The main reason why least-squares methods are more susceptible to loss of mass conservation than Galerkin methods is that they are based on minimization of a functional which includes the continuity equation. In contrast to Galerkin-type methods the mass conservation, i.e. $\nabla \cdot \mathbf{u} = 0$ is a constraint. Because of this, the continuity equations play a different role in the least-squares formulation from the role it plays in Galerkin. Thus, it is clear why least-squares methods are more susceptible to loss of mass conservation than "direct methods".

One way of overcoming the problem of the LSFEM is using the so called restricted LSFEM, see [29], which is based on the least-squares functional with the extension of mass conservation $\nabla \cdot \mathbf{u} = 0$.

Proot and Gerritsma have shown in [18,20] that the Least-Squares Spectral Element Method (LSSEM) leads to good results for such flow problems, since the LSSEM compensate the loss of mass conservation by a superior conservation of the momentum equations for the stationary Stokes equations.

Kattelans and Heinrichs have shown for the stationary Stokes equations in [12] that the LSSCM leads to improved conservation of mass and momentum for internal flow problems. The main reasons for their improved results were that the domain was decomposed into only a few elements, the transfinite mapping of Gordon and Hall was used for discretization, the Clenshaw–Curtis quadrature rule was used for the additional pressure integral condition and the resulting overdetermined algebraic systems were solved by *QR* decomposition.

In this paper we continue the study in [12] for the time-dependent Stokes equations and for the Navier–Stokes equations and we will show that the LSSCM leads to improved mass and momentum conservation for this equations, too.

Furthermore, our approach has the following advantages:

- equal order interpolation polynomials can be employed
- it is possible to vary the polynomial order from element to element
- improved stability properties for small perturbation parameters in singular perturbation problems, [5] and Stokes or Navier–Stokes equations [6–9,11]
- improved conservation of mass and momentum for the stationary Stokes equations, [12]
- good performance in combination with domain decomposition techniques
- direct and efficient iterative solvers for positive definite systems can be used
- implementation is straightforward.

The paper is organized in the following way. In Section 2, the internal flow problem is described. Section 3 introduces the first-order formulation of the Stokes equations and the Navier–Stokes equations. The LSSCM and the discretization are presented in Section 4. The numerical results of our simulations are discussed in Section 5, where we present our results for the time-dependent Stokes equations in Section 5.1 and for the Navier–Stokes equations in Section 5.2. The conclusion is presented in Section 6.

2. The problem set-up

In order to investigate the mass and momentum conservation of the LSSCM we use the same test case as in [29,18,20,12]. The flow problem is defined by a cylinder of diameter d which moves at a speed of one along the centerline of a channel of width $h = 1.5$, see Fig. 1.

The domain of the channel is defined as a rectangle and the center of the cylinder is located at the origin, i.e. we solve the partial differential equations on the domain

$$\Omega_r := \Omega_c \setminus K_r,$$

where $\Omega_c := [-1.5, 3] \times [-0.75, 0.75]$ and $K_r := \left\{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 < r^2\right\}$.

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