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Parameterized preconditioning for generalized saddle point problems arising from the Stokes equation☆

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ABSTRACT

A parameterized preconditioning framework is proposed to improve the conditions of the generalized saddle point problems. Based on the eigenvalue estimates for the generalized saddle point matrices, a strategy to minimize the upper bounds of the spectral condition numbers of the matrices is given, and the explicit expression of the quasi-optimal preconditioning parameter is obtained. In numerical experiment, parameterized preconditioning techniques are applied to the generalized saddle point problems derived from the mixed finite element discretization of the stationary Stokes equation. Numerical results demonstrate that the involved preconditioning procedures are efficient.

1. Introduction

Generalized saddle point problems arise from many scientific and engineering applications, such as mixed finite element methods, constrained optimization, constrained least square problems, image processing, optimal control and so on (see [1]), and usually generate the linear systems in the following form:

$$\begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

the coefficient matrix

$$W = \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \tag{1.2}$$

is called the generalized saddle point matrix, where $A \in \mathbb{R}^{m\times m}$ is symmetric and positive definite, $C \in \mathbb{R}^{n\times n}$ is symmetric and semi-positive definite, $B \in \mathbb{R}^{m\times n},\ m \geq n$, and the Schur complement matrix $S = C + B^T A^{-1} B$ is positive definite (see [1,2]).

A large amount of research work has been devoted to the iterative methods for solving the large scale saddle point problems. Based on the splitting of the matrix W, researchers have developed various stationary iterative methods such

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as Arrow–Hurwicz and Uzawa iterations (see [3]), the inexact Uzawa methods (see [3–6]), the generalized SOR methods (see [7–11]), the HSS method (see [12–14]) and so on. These methods have simple schemes and they are suitable to the parallel computation. Meanwhile, Krylov subspace methods usually have a high efficiency. The two-step CG method, the QMR method, the MINRES method and the GMRES method are introduced to solve the system (1.1) (see [1,3,15–19]). Specially, the MINRES method and the SYMMLQ method (see [19]) cater to the symmetric and indefinite systems and naturally become the candidate of the solvers of system (1.1).

In applications, all of these methods need efficient preconditioners to accelerate the convergence rates. However, establishing a practical preconditioner is usually difficult since the preconditioner is expected to have not only a significant efficiency but also a small computational cost and a clear mechanism. Besides, numerical computing experience teaches us that no method is omnipotent and each type of preconditioner has its own applicability. The main effort of this paper is building a simple and clear preconditioner that can improve the condition of the system (1.1) arising from some applications, such as the mixed finite element method for the stationary Stokes equation. Building such a preconditioner generally depends on the eigenvalue estimates for the generalized saddle point matrices. Fortunately, recently several references have studied the spectral properties of the generalized saddle point matrices (see [2,20–25]), which we believe very important and helpful to establishing the efficient preconditioners.

In this paper, we study the strategy of parameterized preconditioning for the generalized saddle point problems. We attempt to multiply the submatrices by some parameters to precondition the system (1.1). In the theoretical analysis, we give the upper bounds of the spectral condition numbers of the generalized saddle point matrices. Via a primary derivation we minimize the upper bounds of the spectral condition numbers, and then obtain the explicit expression of the quasi-optimal preconditioning parameters. The parameters can be adjusted to the optimum point so that the conditions of the systems may be improved significantly.

The remainder of this paper is organized as follows. In Section 2, we study the framework of parameterized preconditioning and obtain the quasi-optimal choice of the preconditioning parameters, and then give the corresponding preconditioning procedure. In Section 3, based on the different eigenvalue estimates we respectively give two types of preconditioning procedures for the special case $C = O$. In Section 4, we apply the parameterized preconditioning techniques to the systems derived from the mixed finite element discretization of the stationary Stokes equation, and present the numerical results. Finally in Section 5, we give our conclusions.

2. Main results

It is well-known that the smaller condition number of the system may bring the more efficient solution. For instance, for the MINRES method we have the following result (see [19, p 56]):

$$\|r^{(m)}\| \le \frac{1}{T_{[m/2]}(\theta_W)}\|r^{(0)}\|, \tag{2.1}$$

where $r^{(m)}$ is the mth iteration residual and $T_m(\lambda)$ is Chebyshev polynomial of m order, and

$$\theta_W = \frac{\kappa^2+1}{\kappa^2-1},$$

$[\cdot]$ is the integer function, and κ denotes the spectral condition number of the matrix W, which is the coefficient matrix of the linear system. Inequality (2.1) implies that the MINRES method converges faster if the spectral condition number κ becomes smaller.

In the following discussion, we usually use notation $\lambda_i(\cdot)$ to represent the ith eigenvalue of the corresponding matrix, and specially the largest eigenvalue is denoted by $\lambda_1(\cdot)$. We also use notation $\kappa(\cdot)$ to represent the spectral condition number of the corresponding matrix.

It is evident that the matrix W defined by (1.2) can be factorized as

$$W = \begin{pmatrix} I_m & O \\ B^T A^{-1} & I_n \end{pmatrix}\begin{pmatrix} A & O \\ O & -S \end{pmatrix}\begin{pmatrix} I_m & A^{-1}B \\ O & I_n \end{pmatrix},$$

which implies that W is symmetric and indefinite. Hence all the eigenvalues of W are real numbers and located on both sides of the origin. According to the definition of the spectral condition number, it is easy to get the following result.

Lemma 2.1. *Let H be a symmetric and indefinite matrix, and all the eigenvalues of H are located in the interval $I \equiv [l^-, r^-]\bigcup[l^+, r^+]$, where $l^- < r^- < 0 < l^+ < r^+$, then we have*

$$\kappa(H) \le \frac{\max\{r^+, -l^-\}}{\min\{l^+, -r^-\}} = \max\left\{\frac{r^+}{l^+}, \frac{r^+}{-r^-}, \frac{-l^-}{l^+}, \frac{l^-}{r^-}\right\}.$$