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A posteriori estimates of inverse operators for initial value problems in linear ordinary differential equations

T. Kinoshita [a,*], T. Kimura [b], M.T. Nakao [b]

[a] Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

[b] Sasebo National College of Technology, Nagasaki 857-1193, Japan

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ABSTRACT

We present constructive a posteriori estimates of inverse operators for initial value problems in linear ordinary differential equations (ODEs) on a bounded interval. Here, "constructive" indicates that we can obtain bounds of the operator norm in which all constants are explicitly given or are represented in a numerically computable form. In general, it is difficult to estimate these inverse operators a priori. We, therefore, propose a technique for obtaining a posteriori estimates by using Galerkin approximation of inverse operators. This type of estimation will play an important role in the numerical verification of solutions for initial value problems in nonlinear ODEs as well as for parabolic initial boundary value problems.

1. Introduction

In this paper, we consider the positive constant C_{L^2,L^p} in a posteriori estimates of the form

$$\left\| \left(A\frac{d}{dt} + B \right)^{-1} \right\|_{\mathcal{L}(L^2(J)^n, L^p(J)^n)} \leq C_{L^2,L^p}, \tag{1}$$

where $J := (0, T) \subset \mathbb{R}$, $(T < \infty)$ is a bounded interval, n is a positive integer, A is a symmetric positive definite matrix in $\mathbb{R}^{n,n}$, B is an element of $L^\infty(J)^{n,n}$, and p is an arbitrary constant that satisfies $2 \leq p \leq \infty$. For arbitrary $f \in L^2(J)^n$, we consider the following initial value problems in linear ordinary differential equations (ODEs):

$$\begin{cases} Au' + Bu = f, \quad \text{in } J, & \text{(a)} \\ u(0) = 0, & \text{(b)} \end{cases} \tag{2}$$

where $u(t) = (u_1(t), \ldots, u_n(t))^T$ and $\cdot^T$ means transpose. Thus, the problem of estimating (1) and that of estimating the solution u of Eqs. (2)(a) and (b) become equivalent.

In the case of $n = 1$, the solution of (2)(a) and (b) is explicitly written by

$$u(t) = \frac{1}{A} e^{-\frac{1}{A}\int_0^t B(s)\,ds} \int_0^t (e^{\frac{1}{A}\int_0^s B(r)\,dr}) f(s)\,ds. \tag{3}$$

* Corresponding author.
E-mail addresses: kinosita@kurims.kyoto-u.ac.jp (T. Kinoshita), tkimura@post.cc.sasebo.ac.jp (T. Kimura), mtnakao@post.cc.sasebo.ac.jp (M.T. Nakao).

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From the Schwarz inequality, we have

$$|u(t)| \le \frac{1}{A} e^{-\frac{1}{A}\int_0^t B(s)\,ds} \int_0^t (e^{\frac{1}{A}\int_0^s B(r)\,dr})|f(s)|\,ds$$
$$\le \frac{1}{A} e^{-\frac{1}{A}\int_0^t B(s)\,ds} \|e^{\frac{1}{A}\int_0^s B(r)\,dr}\|_{L^2(0,t)} \|f\|_{L^2(0,t)}.$$

Then, we obtain the a priori estimates of u by

$$\|u\|_{L^p(J)}^p \le \frac{1}{A^p} \int_J e^{-\frac{p}{A}\int_0^t B(s)\,ds} \|e^{\frac{1}{A}\int_0^s B(r)\,dr}\|_{L^2(0,t)}^p \,dt \|f\|_{L^2(J)}^p$$

$$\left\| \left(A\frac{d}{dt} + B \right)^{-1} \right\|_{\mathcal{L}(L^2(J),L^p(J))} \le \frac{1}{A} \left(\int_J e^{-\frac{p}{A}\int_0^t B(s)\,ds} \|e^{\frac{1}{A}\int_0^s B(r)\,dr}\|_{L^2(0,t)}^p \,dt \right)^{\frac{1}{p}}. \tag{4}$$

Thus, we can obtain a constant C_{L^2,L^p} that satisfies (1) for $n = 1$. However, the value of C_{L^2,L^p} often increases in these a priori estimates. We define the Galerkin approximate operator of $(A\frac{d}{dt} + B)^{-1}$ and propose a technique for obtaining a posteriori estimates of (1) that are expected to be smaller than (4). In the general case of n, the solution u cannot be written explicitly such as in (3). On the other hand, our method can obtain a posteriori estimates of $\|u\|_{L^p}$ for a general integer n.

On the other hand, the present result can also be applied to the a posteriori estimate for a solution of the following linear parabolic initial boundary value problems

$$\begin{cases} \mathcal{L}_t w \equiv \dfrac{\partial w}{\partial t} - \nu \triangle w + (b \cdot \nabla) w + cw = g, \quad \text{in } \Omega \times J, & \text{(a)} \\ w(x,t) = 0, \quad \text{on } \partial\Omega \times J, & \text{(b)} \\ w(x,0) = 0, \quad \text{in } \Omega, & \text{(c)} \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded convex polygonal or polyhedral domain, $J = (0, T) \subset \mathbb{R}$ a bounded interval, ν a positive parameter, $b \in L^\infty(J; L^\infty(\Omega))^d$, $c \in L^\infty(J; L^\infty(\Omega))$ and $g \in L^2(J; L^2(\Omega))$.

Actually, the technique in this paper can be effectively used to get the estimates of the norm for $\mathcal{L}_t^{-1}$ (see [1] for details).

In Section 2, we introduce some functional spaces and the finite element space and calculate constructive a priori error estimates of the finite element approximation. In Section 3, we propose a posteriori estimates of (1). In Section 4, we show a posteriori error estimates for the exact solution of (2)(a) and (b) and its finite element solution. Here, a posteriori error estimates refer to operator norm error estimates for integral operators. Namely, this error estimate can be calculated for the given finite element space, which is independent of f. Section 5 shows several numerical results.

2. Finite element space

In this section, we introduce functional spaces, projections onto finite dimensional subspaces, and associated error estimates. For $0 < T < \infty$, let J be a finite open set of $\mathbb{R}$, which is defined by $J := (0, T)$. J is divided into m_e subintervals. Let $t_i \in \bar{J}$ be the nodal points satisfying $0 = t_0 < t_1 < \cdots < t_{m_e} = T$. Let each element be represented as $J_i := (t_{i-1}, t_i)$. We define the element size by $|J_i| := t_i - t_{i-1}$ and denote the mesh size by $k := \max_{1 \le i \le m_e} |J_i|$.

2.1. Constructive a priori error estimates for scalar functions

Let $\tilde{S}(J_i, N_i)$ be a finite dimensional subspace of $H_0^1(J_i)$ that depends on the parameter N_i. For example, N_i is the polynomial degree when we employ the finite element method. We define the H_0^1-projection on J_i, $\tilde{P}_{h_i} : H_0^1(J_i) \to \tilde{S}(J_i, N_i)$ by

$$(u - \tilde{P}_{h_i} u, v_{h_i})_{H_0^1(J_i)} = 0, \quad \forall v_{h_i} \in \tilde{S}(J_i, N_i),$$

where $(\cdot,\cdot)_{H_0^1(J_i)}$ is the inner product of Hilbert space $H_0^1(J_i)$, which is defined by $(u, v)_{H_0^1(J_i)} = (u', v')_{L^2(J_i)}$. In this paper, we assume that the following assumption about $\tilde{P}_{h_i}$ holds.

Assumption 2.1. There exists a positive constant $C(|J_i|, N_i) > 0$ that satisfies

$$\|u - \tilde{P}_{h_i} u\|_{H_0^1(J_i)} \le C(|J_i|, N_i) \|u''\|_{L^2(J_i)}, \quad \forall u \in H_0^1(J_i) \cap H^2(J_i), \tag{6}$$

$$\|u - \tilde{P}_{h_i} u\|_{L^2(J_i)} \le C(|J_i|, N_i) \|u - \tilde{P}_{h_i} u\|_{H_0^1(J_i)}, \quad \forall u \in H_0^1(J_i). \tag{7}$$

Assumption 2.1 is the most basic error estimate in the finite element method. For example, in the case of linear polynomial approximation of $H_0^1(J_i)$, the value $C(|J_i|, 1)$ of Assumption 2.1 is obtained by $C(|J_i|, 1) = \frac{|J_i|}{\pi}$. In the case of quadratic polynomial approximation, Assumption 2.1 is satisfied by $C(|J_i|, 2) = \frac{|J_i|}{2\pi}$. Moreover, these constants are optimal constants [2]. In the case of N_i degree polynomial approximation, Assumption 2.1 is satisfied by $C(|J_i|, N_i) = O(|J_i| N_i^{-1})$. However, the optimal constants in this case are not known [3].