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Sensitivity analysis for active control of the Helmholtz equation

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ABSTRACT

In previous works we considered the Helmholtz equation with fixed frequency k outside a discrete set of resonant frequencies, where it is implied that, given a source region $D_a \subset \mathbb{R}^d$ ($d = \overline{2,3}$) and u_0, a solution of the homogeneous scalar Helmholtz equation in a set containing the control region $D_c \subset \mathbb{R}^d$, there exists an infinite class of boundary data on ∂D_a so that the radiating solution to the corresponding exterior scalar Helmholtz problem in $\mathbb{R}^d \setminus D_a$ will closely approximate u_0 in D_c. Moreover, it will have vanishingly small values beyond a certain large enough "far-field" radius R.

In this paper we study the minimal energy solution of the above problem (e.g. the solution obtained by using Tikhonov regularization with the Morozov discrepancy principle) and perform a detailed sensitivity analysis. In this regard we discuss the stability of the minimal energy solution with respect to measurement errors as well as the feasibility of the active scheme (power budget and accuracy) depending on: the mutual distances between the antenna, control region and far field radius R; value of the regularization parameter; frequency; location of the source.

1. Introduction

The current literature has significantly addressed the idea of active manipulation of vector and scalar fields in desired regions of space.

In fact the active (partial) nulling of fields in the low-frequency acoustics was first studied in [21] (feed-forward control of sound) and in [28] (feedback control of sound). The reviews [33,15,19,20,32,9,31,7] and references therein discuss the active sound control problem for arbitrary finite frequency.

For cloaking applications, the strategy proposed in [26] employs a continuous active layer on the boundary of the control region while the scheme discussed in [11–14,10] (see also [37]), uses a discrete number of active sources located in the exterior of the control region to manipulate the fields. These field manipulation problems may be thought of as a prelude to the recommended line of research proposed here. In these works, manipulation of electrostatic fields has been treated as well as time varying scalar fields in two and three dimensions by using integral representation theorems. These results were further extended in [27], where a detailed sensitivity analysis is presented.

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In a parallel development in [34,35] the authors make use of the equivalence principle and show how a dipole array can cancel the known electromagnetic scattered field from a cylinder. Further experimental results on active cloaking are presented in [22] for the quasistatic regime and in [6] for the finite frequency regime.

Yet another approach for active cloaking is the use of essentially non-radiating sources. The works [24,23] and [25] studied the structure of such sources and in [5,4] the author considered their applications for the active cloaking.

In [2] an alternative active scattering cancellation strategy is proposed where it is shown that suitably designed active meta-surfaces (e.g., active smart electrical circuits) can in principle be tuned to suppress the scattering from known incident fields. These results are supported by simulation and in [36] experimental proof was obtained for a cylindrical scatterer of 2.2-wavelengths in length and 0.31-wavelengths in cross section. In all the above works foreknowledge of the fields to be reduced was available.

The active manipulation of quasistatic fields by using one source was studied with the help of boundary layer potentials in [29]. Recently in [30] we extended the results presented in [29] to the active control problem for the exterior scalar Helmholtz equation. In particular, we characterized an infinite class of boundary functions on the source boundary ∂D_a so that we achieve the desired manipulation effects in several mutually disjoint exterior regions. The method is novel in the sense that instead of using microstructures, exterior active sources modeled with the help of the above boundary controls are employed for the desired control effects. Such exterior active sources can represent velocity potential, pressure or currents depending on the regime of interest.

In the current paper we propose a sensitivity and feasibility study for the minimal norm solution of the active manipulation problem considered in [29,30]: one antenna D_a approximating a given field in a prescribed near field control region D_c with very little radiation on $\partial B_R(\mathbf{0})$, with $R \gg 1$ (see Fig. 1). We make use of the results in [30] and present a detailed sensitivity and feasibility study for the minimal norm solution of the problem. As we will explain in Section 2 this problem may be relevant to protection from unwanted interrogation as well as for the question of near field synthesis with small far field radiation.

The paper is organized as follows: In Section 2 we present the physical motivation behind our study. In Section 3 we recall the general result obtained in [30] in the context of exterior active cloaking. Section 4 we present an L^2 conditional stability result for the minimal norm solution with respect to measurement errors of the incoming field. In Section 5 we present the numerical details of the Tikhonov regularization algorithm with the Morozov discrepancy principle for the computation of the minimal norm solution of the exterior active cloaking problem in two dimensions. We will numerically observe the fact that the scheme requires large antenna powers in the far field and we will provide numerical support for our theoretical stability results. An important part of this section will be focused on the sensitivity analysis, where we will study: the dependence of the control results as a function of mutual distances between the antenna, control region and far field region; and the broadband character of our scheme in the near field region. Finally, in Section 6 we highlight the main results of the paper and discuss current and future challenges and extensions of our research.

2. Motivation and main results

The motivation behind our work is the desire to create stable, low budget schemes, for the approximation of desired fields in the exterior of controlled active sources with possible applications in antenna synthesis, inverse source problems, and electromagnetic or acoustic cloaking/shielding.

Our analysis focuses on a the following situation:

Problem. A single active antenna, D_a, approximates a desired pattern in its near field region, D_c, with very little spillover in the far field (e.g. beyond a certain fixed radius R).

The geometry of the problem is sketched in Fig. 1. In Proposition 4.1 and Corollary 4.1 we prove, under suitable source type conditions, the stability of the minimal energy solution (the Morozov solution) of the above problem. In Section 5 we also provide a sensitivity numerical analysis of the minimal energy solution (e.g., its stability, power budget and accuracy) with respect to various parameters: distance between antenna D_a and control region D_c, wave number k, noise level ϵ, far field boundary R. The numerical sensitivity analysis performed in Section 5 suggest that the scheme proposed by us has a broadband character and is feasible (good accuracy with good power budget) only in a thin sub region in the near field of the sources but possible for a wide enough angular span.

Our result proposes a stable scheme for the synthesis of active sources with controllable near fields and very weak far fields and, besides their possible applicability in near field synthesis applications, this type of sources are theoretically very important in the analysis of the inverse source problems since they are near the kernel of the far field operator and thus are the main cause of instabilities. Their understanding will guide us to a proper penalization of the cost functional to avoid such instabilities. A separate publication with the 3D analysis is in preparation and will be communicated soon.

The presented results imply that one can control a thin but angularly wide near field region of a small active weak radiator and this fact implies the possibility to build a suitable array of such elements with the property that they will cancel an incoming field with very little radiation in the far field. Thus, paired with an appropriate feedback control loop,