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Compensation of domain modelling errors in the inverse source problem of the Poisson equation: Application in electroencephalographic imaging

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A B S T R A C T

In the inverse source problem of the Poisson equation, measurements on the domain boundaries are used to reconstruct sources inside the domain. The problem is an ill-posed inverse problem and it is sensitive to modelling errors of the domain. These errors can be boundary, structure and material property errors, for example. In this paper, we investigate whether the recently proposed Bayesian approximation error (BAE) approach could be used to alleviate the source estimation errors when an approximate model for the domain is employed. The BAE is based on postulating a probabilistic model for the uncertainties, in this case the geometry and structure of the domain, and to carry out approximate marginalization over these nuisance parameters. We particularly consider electroencephalography (EEG) source imaging as an application. EEG is a diagnostic brain imaging modality, and it can be used to reconstruct neural sources in the brain from electric potential measurements along the scalp. In the feasibility study, we assess to which degree one can recover from the modelling errors that are induced by the use of the three concentric circle head model instead of an anatomically accurate head model. The studied domain modelling errors include errors in the geometry of the exterior boundary and the structure of the interior. We show that, in particular with superficial dipole sources, the BAE yields estimates that can in some cases be considered adequately accurate. This would avoid the need for the extraction of the accurate head features which is conventionally carried out via expensive and time consuming auxiliary imaging modalities such as magnetic resonance imaging.

1. Introduction

In the inverse source problem of the Poisson equation, the goal is to reconstruct sources inside the computational domain based on measurements around its boundaries. The problem has many practical applications such as bioluminescence tomography [14], and electroencephalographic [12] and magnetoencephalographic imaging [13]. The inverse source problem is an ill-posed inverse problem, and stable estimates cannot be computed from noisy boundary measurements without ei-

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ther regularization or by employing prior models. Moreover, the inverse solution is highly sensitive to modelling errors in addition to measurement errors. Typical modelling errors are related to the geometry, internal structure and material properties of the domain. Because such errors are commonly encountered in brain imaging, we consider electroencephalography (EEG) source imaging as a potential application in this paper.

EEG is a routinely used brain imaging modality to diagnose, for example, epilepsy. In EEG source problem, neural sources are reconstructed inside the brain based on electric potential measurements around the scalp, and it is well known that the inverse solution depends strongly on the accuracy of discretized head geometry [2,4,7,8,16,39,49,54] and the accuracy of electric conductivity modelling of different tissues [3,47,25,48,34,50,51]. The head features can be extracted, to some extent, by using multi-modal imaging (computed tomography/diffusion magnetic resonance imaging), for example. However, such imaging is an expensive task and requires robust image segmentation, registration and post-processing algorithms [42]. Therefore, approaches that allowed the use of approximate head models would be highly desirable.

In this paper, we investigate to which extent the domain modelling errors can be compensated by considering the Bayesian approximation error (BAE) approach which was introduced in [19]. BAE is applied solely to the discretized problem with finite number of boundary measurements, and the idea is to use an approximate model for the domain and to estimate the statistics of the errors between the approximate and accurate models. These errors are induced by a postulated probabilistic model for the uncertainties, in this case, the domain geometry and structure. In the resulting overall observation model, the modelling errors appear as additive errors which are then marginalized using a Gaussian approximation.

Previously, BAE has been shown to be a versatile and efficient approach to compensate for various approximation and modelling errors, for example, in electrical impedance tomography (EIT) [24,26,30,31] and diffuse optical tomography (DOT) [1,22,23,29,44,45]. In particular, EIT was employed for the reconstruction of the conductivity distributions in the thorax, and the approximation errors related to the use of a generic geometry (cylinder) instead of an actual thorax with severe model reduction were handled with BAE in [31]. Similarly in DOT, the effects of mismodelled head geometry were compensated with BAE in [29]. Thus, the BAE has been shown to be successful in compensating errors related to the geometric mismodelling in diffuse tomographic problems.

In this paper, in contrast to diffuse tomographic problems, we are not interested in the material properties of the domain but a vector-valued source field that, however, depends on the modelling of the domain. In addition to boundary mismodelling, we also consider modelling errors related to the interior of the domain which was not done in [31,29]. In the EEG source imaging examples, we consider only focal dipole sources, that are common in epilepsy, and the case of finite number of boundary measurements (instead of continuous boundary data). In order to recover focal sources, we employ sparsity promoting priors rather than Gaussian smoothing priors as in [31,29].

We use an anatomical atlas to construct the model for the distribution of the head geometry and structure. Because the head can be segmented into various number of different tissues, we consider here the three and five compartment models as the anatomically accurate domain models (head conductivity models). Then the three concentric circle model is used as the approximate head model. To access the ground truth, we use simulations in this feasibility study. We also consider both noiseless and noisy data to isolate the effects of domain modelling errors, and to assess the practical case in which both modelling and measurement errors are present. When successful, the use of BAE would thus allow the same generic head model to be used for all patients.

2. Theory

In this section, we define the forward models used in this paper, give a brief review of the Bayesian framework for inverse problems, and the Bayesian approximation error approach. More information about the Bayesian framework in general can be found in, for example, [6,19,43] and the approximation error approach in particular in [1,19,18,17].

2.1. Poisson equation

The computational domain is denoted with Ω and its material properties with $\sigma(x)$ where $x \in \Omega$. The Poisson equation has the form

$$\nabla \cdot \sigma(x)\nabla u(x) = f(x), \quad x \in \Omega, \tag{1}$$

where u is the scalar potential and $f(x)$ is the source term. The boundary conditions are

$$u = 0, \quad x \in S_{\mathrm{D}} \tag{2}$$

$$\sigma(x)\frac{\partial u}{\partial \hat{n}} = 0, \quad x \in S_{\mathrm{N}}, \tag{3}$$

where $\partial\Omega = S_{\mathrm{D}} \cup S_{\mathrm{N}}$ is the boundary, $S_{\mathrm{D}} \cap S_{\mathrm{N}} = \varnothing$, and $\hat{n}$ is the unit normal vector of the boundary.

In case of EEG imaging, the Poisson equation can be used under the quasi-static approximation of the Maxwell's equations. Furthermore, in EEG the source term is often of the form $f(x) = \nabla \cdot d(x)$ where $d(x) : \Omega \mapsto \mathbb{R}^k$ is a vector valued function that describes the neural sources as idealized electric dipoles [41]. Here, k is equal to either 2 or 3 depending whether the analysis is carried out in 2D or 3D.