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On the robustness of ILU smoothers on triangular grids

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ABSTRACT

In this work, incomplete factorization techniques are used as smoothers within a geometric multigrid algorithm on triangular grids. A local Fourier analysis is proposed to study the smoothing properties of these methods, as well as the asymptotic convergence of the whole multigrid procedure. With this purpose, two- and three-grid local Fourier analysis are performed. Several two-dimensional diffusion problems, including different kinds of anisotropy are considered to demonstrate the robustness of this type of methods.

1. Introduction

The solution of large sparse algebraic systems of equations is of great interest due to the wide range of applications in which they are involved, as they occur for example in the numerical solution of partial differential equations. Since the factor matrices L and U do not have a sparse structure anymore, to compute the exact LU factorization of a large but sparse $n \times n$ regular matrix $\mathcal{O}(n^3)$ arithmetic operations are needed, which is not practicable for large values of n. Incomplete factorization methods can be seen as approximation of the exact LU factorization of a matrix A in which fill-ins are allowed only at a restricted set of positions in the LU factors, in order to ensure that the action of A^{-1} is inexpensive. These iterative methods can be applied to any sparse matrix in principle and they are often very robust with respect to anisotropic coefficients for example.

The class of incomplete factorization methods was first introduced in [5,19,20] as methods for solving discretizations of PDEs. However, they can be seen in a more general way as matrix splitting techniques, see [28,29]. Furthermore, since Evans in [8] proposed the use of sparse LU factors as preconditioners, these methods have been successfully applied as preconditioners for linear systems. Meijerink and Van der Vorst [17] proved the existence of incomplete LU (ILU) preconditioners for M-matrices, and Gustafsson [10] modifies the incomplete factorization preconditioner in order to obtain improved spectral properties. A good introduction to these methods can be found in the books by Hackbusch [12] and Axelsson [2], and in the overview papers [1,7], for example. Finally, in [13], a collection of research papers related to incomplete decompositions is presented.

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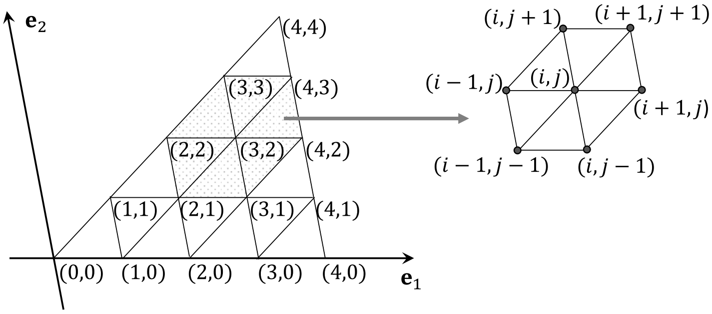

Fig. 1. New basis in $\mathbb{R}^2$ fitting the geometry of the triangular grid, local numbering of the regular grid obtained after two refinement levels and numbering for the stencil operator.

Multigrid methods [11,25,26,31] aim at improving the convergence of classical iterative methods for solving large sparse systems of algebraic equations. They are among the fastest solvers for this kind of problems. However, they strongly depend on the so-called components of the algorithm. In geometric multigrid, usually the most influent component is the relaxation procedure. ILU decompositions were introduced as smoothers in the multigrid technique by Wesseling et al. (see [30,32]), and since then, they were often used as smoothers for different applications (see [15,24,35,36], for example). Also, a modified ILU smoother was proposed in [16,18,23,36], which has favorable properties for dealing with anisotropic problems. In particular, G. Wittum [36] was the first to rigorously prove the robustness of this smoother for an anisotropic model problem, and R. Stevenson [23] generalized the results in [36]. More recently, in [27], a fast preconditioner solver based on multigrid with an ILU smoother was proposed for solving heterogeneous high-wavenumber Helmholtz problems, and in [21], a new relaxation methodology based on a truncated ILU smoother is presented for multigrid preconditioning of discrete convection–diffusion problems.

Local Fourier analysis (LFA) [3,4,33] deals with the quantitative analysis of geometric multigrid methods. It is used to analyze the smoothing properties of the relaxation procedures or/and the asymptotic convergence of the two-, three-, ..., k-grid methods. LFA provides very accurate predictions of the asymptotic convergence factors of these algorithms for many problems, especially for elliptic problems, and for this reason it becomes a very practical tool for the design of suitable multigrid components for different problems. This analysis assumes some simplifications: boundary conditions are neglected by formally defining the discrete operator, which must be given by a constant stencil, on an infinite regular grid. Regarding the application of LFA for the study of incomplete factorization techniques, one of these assumptions is not satisfied, since near the boundaries of the domain the stencils involved in the decomposition vary. However, they tend rapidly to constant stencils away from the boundaries and therefore local Fourier analysis is able to provide useful predictions also for this type of methods. For example, its usefulness is proven in [31], where a smoothing analysis for a wide variety of incomplete factorization methods applied to different problems on rectangular grids was performed. Smoothing properties of ILU-type smoothers on rectangular grids were studied in [6,15,35,36], for example. However, the analysis of these relaxation processes on triangular grids, as well as a more detailed study of the multigrid convergence (two-, three- or k-grid analysis) is missing in the literature. Therefore, in this work, we want to close this gap in order to provide a tool to study these methods in a more flexible framework.

The remainder of the paper is organized as follows. In Section 2, the considered incomplete factorization techniques on triangular grids are introduced. Section 3 shows how to perform a local Fourier analysis for multigrid based on ILU smoothers. Smoothing as well as two- and three-grid analysis are described. In Section 4, local Fourier analysis results are displayed for different two-dimensional diffusion equations, including also anisotropic problems. In this section, we demonstrate the sharpness of the proposed LFA, showing a very accurate match between theoretical and experimentally computed convergence factors. Finally, in Section 5, we present two numerical experiments to show on the one hand the suitability of ILU smoothers for some anisotropic problems and on the other hand to illustrate the usefulness of the LFA to build block-wise multigrid methods on semi-structured triangular grids.

2. Incomplete decomposition techniques for structured triangular grids

We are interested in the study of incomplete decomposition techniques for symmetric matrices arising from the discretization of partial differential equations on structured triangular grids. These meshes arise from the application of a regular refinement process to a triangle, that is, the regular triangular grid is constructed by dividing the triangle into four congruent triangles connecting the midpoints of the edges, and so forth until the mesh has the desired fine scale to approximate the solution of the problem. In this way, for a fixed number of refinement steps ℓ, we can define the regular grid arising inside the triangle as follows

$$G_\ell = \{\mathbf{x} = (x, y)|_{\{\mathbf{e}_1,\mathbf{e}_2\}} \mid x = k_x h_x,\ y = k_y h_y,\ k_x = 0, \ldots, 2^\ell,\ k_y = 0, \ldots, k_x\}, \tag{2.1}$$