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# Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography

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ABSTRACT

We propose the generalized quadrature methods for numerical solution of singular integral equation of Abel type. We overcome the singularity using the analytic computation of the singular integral. The problem of solution of singular integral equation is reduced to nonsingular system of linear algebraic equations without shift meshes techniques employment. We also propose generalized quadrature method for solution of Abel equation using the singular integral. Relaxed errors bounds are derived. In order to improve the accuracy we use Tikhonov regularization method. We demonstrate the efficiency of proposed techniques on infrared tomography problem. Numerical experiments show that it makes sense to apply regularization in case of highly noisy (about 10%) sources only. That is due to the known fact that Volterra equations of the first kind enjoy selfregularization property.



## 1. Introduction

Numerical methods for solving a variety of singular integral equations (SIE) are offered in many publications, here readers may refer to [2–5,13,14,18,21,22,24,27,31–34,37,40,41] and others. A one-dimensional SIE of the 1st and 2nd kind with the Cauchy and Hilbert kernels, logarithmic et al., as well as two-dimensional, nonlinear SIE have been addressed. In present article we concentrate on Abel singular integral equation [1,2,6–9,14,22,24,26,35–37,39,40]

$$2\int_x^R \frac{r}{\sqrt{r^2-x^2}}k(r)\,dr = q(x), \quad 0 \le x \le R, \tag{1}$$

where $k(r)$ is desired function, $q(x)$ is the source function. Equations (1) are widely used in practical models including plasma diagnostics, infrared tomography, X-ray CT, spectroscopy, star clusters astrophysics, etc. In all these problems the object of interest enjoys the axial (or spherical) symmetry. Abel equation also can be written as

$$\int_0^x \frac{k(r)}{\sqrt{x-r}}\,dr = q(x),\ 0 \le x \le R. \tag{2}$$

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It has been studied in this form in [2,5,13,14,18,21,22,27–29,31,34,39,41]. Equation (2) describes various problems in mechanics (such as tautohron problem), scattering and other problems. Of course, one may transfer SIE (1) into SIE (2) and vice versa, but it makes it more complicated to analyze their physical meaning.

Let us below outline the main algorithms for numerical solution of SIE and singular integrals computation. For more details readers may refer to [36,37].

1. *Algorithms based on relevant mesh shift.* In papers [3,4] the discrete meshes of knots with respect to variables $r$ and $x$ are introduced, i.e. $r_j = jh$, $x_i = r_i + \Delta$, $j, i = 0, 1, \dots, n$, $r_n = R$, where step $h = R/n$, $\Delta$ is mesh shift which is $h/2$ [3] or $\Delta \in (0, h/2)$ [4]. Introduction of the shift $\Delta$ enables singularity overcome when it comes to quadrature rules application. But such algorithms need this shift selection.
2. *Quadrature type methods.* One of the popular methods (here readers may refer to work [3]) is Discrete Vortices Method where the integral with Cauchy kernel

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(x)}{x-x_0}\,dx = f(x_0),\ -1 < x_0 < 1,$$

is approximated with left rectangles quadrature rule and using meshes on $x$ and $x_0$ with shift $\Delta = h/2$. This gives the system of linear algebraic equations (SLAE) with finite main diagonal.
In work [8] the "onion peeling" method for solution of SIE (1) is suggested. Here region $r \in [0, R]$ is approximated with rings $\Delta r$ wide of constant values $k \in (r_j - \Delta r/2, r_j + \Delta r/2)$ for each $r_j$. Here meshes are assumed to be uniform ($\Delta = 0$). The main idea in this method is that integral $\int_{r_j-\Delta r/2}^{r_j+\Delta r/2}\frac{r}{\sqrt{r^2-x_i^2}}\,dr$, $r_j - \Delta r/2 \ge x_i$ is computed analytically and it is finite. Further midpoint quadrature is used resulting systems of linear algebraic equations with upper triangular matrix with respect to $k_j = k(x_j)$. The similar method is suggested in [37].
3. *Solution approximation.* In works [2,24,30,31,34,40] et al., the desired solution $k(r)$ (as well as the right-hand side $q(x)$) is approximated with an orthogonal polynomial, shifted Legendre polynomials, normalized Bernstein polynomials, algebraic or trigonometric polynomial or polynomial spline with coefficients determined with minimum of discrepancy between the left-hand side and right-hand side of (1). This leads to a projection method (the Galerkin method, the collocation method, the method of splines, the quadrature method, the least squares method, etc.) and to the solution of a SLAE with respect to the corresponding polynomial coefficients.
In these algorithms, there is a *self-regularization*, and in the case of using the relative shift of meshes, the shift $\Delta$ plays the role of the regularization parameter. Namely if $\Delta$ is closer to $h/2$ then solution $k(r)$ is more stable, but it makes reduction of resolving capability of the method. If $\Delta$ is closer to zero, then solution is less stable but resolving capability of the method is higher. In all these algorithms, a SLAE is with diagonally dominant matrix (but not infinite).
We also note a number of algorithms. Equation (1), as is known, has an analytical solution [1,2,6,8,24,30,37]

$$k(r) = -\frac{1}{\pi}\int_{r}^{R}\frac{q'(x)}{\sqrt{x^2-r^2}}\,dx, \quad 0 \le r \le R. \tag{3}$$

However, solution (3) contains derivative $q'(x)$ of experimental (noisy) function $q(x)$ and the problem of differentiation is ill-posed [38]. Moreover, integral in (3) is improper (singular). Nevertheless, a number of the following algorithms is proposed to compute the solution according to (3).
4. *Interpolation and quadrature method.* In [6], derivative $q'(x)$ was computed using interpolation on three (and two) neighboring points (discrete values of $x$). Integral $\int_r^R \frac{dx}{\sqrt{x^2-r^2}}$ (cf. (3)) is computed analytically (without singularity). The similar algorithm was suggested in [37] using generalized left rectangles formula.
5. *Approximation of the right-hand side $q(x)$* is used in works [18,24,40]. Function $q(x)$ is suggested to be approximated by a linear combination of smoothing polynomials (or splines) uniform for the whole interval $x \in [0, R]$. Derivative $q'(x)$ is computed using polynomial (or spline) differentiation. Solution $k(r)$ in accordance with (3) is computed by summing the integral in (3) along segments that performed analytically (see [40, pp. 188–189]).
6. *Algorithm without using derivative $q'(x)$.* In [9], formula (3) is converted (by means of integration by parts) into the following expression that does not contain derivative $q'(x)$ (cf. [8,40]):

$$k(r) = -\frac{1}{\pi}\left\{\frac{q(R)-q(r)}{\sqrt{R^2-r^2}} + \int_{r}^{R}\frac{x\,[q(x)-q(r)]}{\sqrt{(x^2-r^2)^3}}\,dx\right\}, \quad 0 \le r \le R.$$

This algorithm is implemented, e.g., in paper [40, pp. 217–220] using the cubic spline (see [23, p. 273]) for $q(x)$.
7. *Use of regularization.* Abel equation is the Volterra equation of the first kind. It enjoys self-regularization property. That is why this is the moderately ill-posed problem [8]. This means that above mentioned algorithms are moderately stable. Nevertheless, in papers [1,7,8,14] et al., the Tikhonov regularization method [10,16,38,39] was used to enhance the stability of algorithms.