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Analysis of numerical solutions to Sommerfeld integral relation of the half-space radiator problem

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A R T I C L E I N F O

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A B S T R A C T

Sommerfeld integrals relate a spherical wave from a point source to a convolution set of plane and cylindrical waves. This relation does not have analytical solutions but it submits to a solution by numerical integration. Among others, it is significant for theoretical studies of many optical and radiation phenomena involving surfaces. This approach is preferred over discretized computational models of the surface because of the many orders of increased computations involved in the latter. One of the most widely used and accurate methods to compute these solutions is the numerical integration of the Sommerfeld integrand over a complex contour. We have analyzed the numerical advantages offered by this method, and have justified the optimality of the preferred contour of integration and the choice of two eigenfunctions used. In addition to this, we have also analyzed four other approximate methods to compute the Sommerfeld integral and have identified their regions of validity, and numerical advantages, if any. These include the high relative permittivity approximation, the short distance approximation, the exact image theory and Fourier expansion of the reflection coefficient. We also finally compare these five methods in terms of their computational cost.

1. Introduction

The problem of modelling radiating antennas near a plane boundary such as the earth–air interface is of considerable practical importance in the fields of nano optics, oceanography, geophysical exploration, and submarine communication and detection. The radiation characteristics of an antenna can be substantially affected by the presence of a lossy ground with finite conductivity [4], especially in the near field of the antenna. Applications in nano optics also require the computation of interactions between a large number of dipole sources and substrates, thus requiring a repetitive use of this method numerous times to obtain a final solution of the problem [10,11,17,24,27,25,6,12,28]. Due to the vastly diminished scales involved in nano optics, the interaction between the dipole sources and the substrate is very significant [26,8,23].

The classic formulation of this problem by Sommerfeld [22,20,21] assumes a homogeneous lossy half-space with finite conductivity (say, the earth), and an infinitesimal vertical point dipole embedded in the free space (say, air) above it. Maxwell's equations are applied subject to the half-space boundary conditions, and the solutions are obtained in the form of an inverse Fourier–Bessel integral. This integral modelling the interaction of the dipole with the surface is known as the Sommerfeld integral. Since the problem has cylindrical symmetry, it is convenient to express the solution in cylindrical coordinates as an integral of the eigenfunctions of the cylindrical Helmholtz operator. These eigenfunctions are in terms

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of the zeroth order Bessel function of the first kind for the ρ direction, and in terms of a complex exponential for the z direction. Because of the azimuthal symmetry in angle ϕ, the solutions are independent of ϕ.

The Sommerfeld integral has no closed form analytic solution, but there exist a few numerical methods to compute the solution. Since the integrand of the Sommerfeld integrand is oscillatory and has branch points along with other singularities, traditional numerical integration schemes converge poorly if the path of integration is not optimally chosen. Approximate analytic expressions can also be obtained when certain constraints are applied on the parameters. The objective of this work is to study the efficiency of the methods proposed so far, to evaluate the Sommerfeld integral in its amenable or approximated forms. Note that for more general scattering problems with arbitrarily shaped scatterers, other efficient integral equation solvers exist [3]. Similarly, these methods specific to this integral are expected to converge fast even if advanced numerical integration schemes designed for various types of singularities are not used. One method – exact image theory – is similar to integration schemes proposed for integrating functions with certain types of singularities [2,9,7,1]. The Sommerfeld integral based approach can in principle be combined with other integral equation solvers for problems involving an infinite surface and other scatterers [19].

The three spatial components of the vector potential for a z oriented oscillating electric point dipole in the near field of a surface are given in (1a) and (1b). The first two terms on the right hand side of (1a) represent the vector potential due to the primary dipole source and the image respectively. The third integral term is known as the Sommerfeld integral.

$$A_z = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} - 2\int_0^\infty J_0(k_\rho \rho) e^{-k_z(z+h)} \frac{k_{zs}}{\varepsilon k_z + k_{zs}} \frac{k_\rho}{k_z} \mathrm{d}k_\rho \tag{1a}$$

$$A_\rho = A_\phi = 0 \tag{1b}$$

$$k_z = \sqrt{k_\rho^2 - k^2} \tag{1c}$$

$$k_{zs} = \sqrt{k_\rho^2 - \varepsilon k^2} \tag{1d}$$

$$R = \sqrt{\rho^2 + (z-h)^2} \tag{1e}$$

$$R' = \sqrt{\rho^2 + (z+h)^2} \tag{1f}$$

where ε is the relative permittivity of the surface, k is the wave number in free space, k_ρ and k_z are the ρ and z components respectively of the wave number in free space, k_{zs} is the z component of the wave number under the surface, ρ and z are the coordinates of the observation point, and J_0 is the zeroth order Bessel function of the first kind.

Note the branch cuts and singularities of this integrand are shown in Fig. 1. We describe five methods to evaluate this integral; their regions of validity in terms of the permittivity of the half-space and the $^\rho/_z$ ratios. Finally, in Section 7, we comment on the approximate computational costs involved in each of these methods.

2. Complex contour integration

For analytical integration, as a consequence of Cauchy's residue theorem, all different complex contours are equivalent provided the closed loop formed by the different contours of integration do not enclose any poles or singularities of the integrand and the contours do not intersect any branch cut of the integrand. However, numerical quadrature schemes would encounter oscillatory or non-oscillatory integrands depending on the contour chosen, and this can significantly affect the rate of convergence and accuracy of the numerical result.

2.1. Contour integration with Bessel functions

We start by ascertaining the location of the poles, branch points and branch cuts of the Sommerfeld integrand. The integrand has a pole p, given by (2), corresponding to $\varepsilon k_z + k_{zs} = 0$.

$$k_\rho = p = k\sqrt{\frac{\varepsilon}{1+\varepsilon}} \tag{2}$$

It also has branch points at $k_\rho = \pm k$ and $k_\rho = \pm k\sqrt{\varepsilon}$ due to $\sqrt{k_\rho^2 - k^2}$ and $\sqrt{k_\rho^2 - \varepsilon k^2}$ respectively. Choosing the conventional principal value of the square root function, the corresponding branch cuts are as given by (3). The detailed derivations which result in these branch cuts are shown in Appendix B.

$$k_\rho = t, \quad t \in [-k, k] \tag{3a}$$

$$k_\rho = it, \quad t \in \mathbb{R} \tag{3b}$$

$$k_\rho = t + i\frac{\varepsilon_y k^2}{2t}, \quad t \in \left[-k\Re\{\sqrt{\varepsilon}\}, k\Re\{\sqrt{\varepsilon}\}\right] \tag{3c}$$