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Analysis of *a posteriori* error estimates of the discontinuous Galerkin method for nonlinear ordinary differential equations

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I would like to dedicate this work to my Father, Ahmed Baccouch, who unfortunately passed away during the completion of this work

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ABSTRACT

We develop and analyze a new residual-based *a posteriori* error estimator for the discontinuous Galerkin (DG) method for nonlinear ordinary differential equations (ODEs). The *a posteriori* DG error estimator under investigation is computationally simple, efficient, and asymptotically exact. It is obtained by solving a local residual problem with no boundary condition on each element. We first prove that the DG solution exhibits an optimal $\mathcal{O}(h^{p+1})$ convergence rate in the L^2-norm when p-degree piecewise polynomials with $p \geq 1$ are used. We further prove that the DG solution is $\mathcal{O}(h^{2p+1})$ superconvergent at the downwind points. We use these results to prove that the p-degree DG solution is $\mathcal{O}(h^{p+2})$ super close to a particular projection of the exact solution. This superconvergence result allows us to show that the true error can be divided into a significant part and a less significant part. The significant part of the discretization error for the DG solution is proportional to the $(p+1)$-degree right Radau polynomial and the less significant part converges at $\mathcal{O}(h^{p+2})$ rate in the L^2-norm. Numerical experiments demonstrate that the theoretical rates are optimal. Based on the global superconvergent approximations, we construct asymptotically exact *a posteriori* error estimates and prove that they converge to the true errors in the L^2-norm under mesh refinement. The order of convergence is proved to be $p+2$. Finally, we prove that the global effectivity index in the L^2-norm converges to unity at $\mathcal{O}(h)$ rate. Several numerical examples are provided to illustrate the global superconvergence results and the convergence of the proposed estimator under mesh refinement. A local adaptive procedure that makes use of our local *a posteriori* error estimate is also presented.

1. Introduction

In this paper, we investigate the superconvergence properties and analyze a residual-based *a posteriori* error estimator of the discretization errors for the discontinuous Galerkin (DG) method applied to the following first-order initial-value problem (IVP) of nonlinear ordinary differential equation (ODE) on $[0, T]$:

$$\frac{d\vec{u}}{dt} = \vec{f}(t, \vec{u}), \quad t \in [0, T], \quad \vec{u}(0) = \vec{u}_0, \tag{1.1}$$

where $\vec{u} : [0, T] \to \mathbb{R}^n$, $\vec{u}_0 \in \mathbb{R}^n$, and $\vec{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$. In our analysis, we assume that the solution exists and is unique. According to the ODE theory, the condition $\vec{f} \in C^1([0, T] \times \mathbb{R}^n)$ is sufficient to guarantee the existence and uniqueness of the

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solution to (1.1). We note that an IVP for higher-order ODE may be solved using the DG method for solving first-order system of the form (1.1) since one can transform higher-order equations into several coupled first-order equations by introducing new unknowns.

ODEs solvers are important tools for the computational solutions of higher-order ODEs and many partial differential equations (PDEs). For instance, the application of the standard finite element method or DG in space to solve time-dependent PDEs generates a coupled system of ODEs. Once the spatial discretization is constructed, one would then need to employ a suitable ODE solver for the time discretization. If the mesh size in space becomes small then often the ODE system becomes more and more stiff. It is very well-known that explicit schemes suffer from extremely small time step restriction for stability. Therefore, explicit time discretization techniques are not a suitable choice and implicit, at least A-stable methods are desirable. There are many A-stable higher-order time discretization schemes designed for different purposes in the literature, such as the implicit Runge–Kutta (IRK) methods and DG methods with higher polynomial order. Despite the popularity of high-order IRK methods for integrating systems of ODEs, we choose the DG method because of the following advantages: (i) it is A-stable, (ii) it is locally conservative, and (iii) it exhibits strong superconvergence that can be used to estimate the discretization error. Furthermore, it is very natural to construct high-order DG methods and, for future developments, we can apply the well-known adaptive DG techniques for changing the polynomial degree as well as the length of the time intervals. Finally, if we use space–time DG methods for the discretization of evolution PDEs, we would have a uniform variational approach in space and time which may provide a useful tool for the future analysis of the fully discrete problem and the construction of simultaneous space–time adaptive methods.

The DG method considered here is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. DG method combines many attractive features of the classical finite element and finite volume methods. It is a powerful tool for approximating some differential equations which model problems in physics, especially in fluid dynamics or electrodynamics. Comparing with the standard finite element method, the DG method has a compact formulation, *i.e.*, the solution within each element is weakly connected to neighboring elements. DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [36]. In 1974, Lesaint and Raviart [33] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ODEs [6,19,32,33], hyperbolic [15–18,23,24,35,29,31,20,45,34,2,3,12,5] and diffusion and convection–diffusion [13,14,43,25] PDEs. The proceedings of Cockburn et al. [22] and Shu [39] contain a more complete and current survey of the DG method and its applications.

In recent years, the study of superconvergence and *a posteriori error* estimates of DG methods has been an active research field in numerical analysis, see the monographs by Verfürth [41], Wahlbin [42], and Babuška and Strouboulis [9]. A knowledge of superconvergence properties can be used to (*i*) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (*ii*) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. *A posteriori* error estimates play an essential role in assessing the reliability of numerical solutions and in developing efficient adaptive algorithms. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [8]. Superconvergence properties for finite element and DG methods for ordinary differential equations have been studied in [6,7,27,33,44,40]. The first superconvergence result for standard DG solutions of hyperbolic PDEs appeared in Adjerid et al. [6]. The authors presented numerical results that show that standard DG solutions of one-dimensional linear and nonlinear hyperbolic problems using p-degree polynomial approximations exhibit an $\mathcal{O}(h^{p+2})$ superconvergence rate at the roots of $(p+1)$-degree Radau polynomial. They further established a strong $\mathcal{O}(h^{2p+1})$ superconvergence at the downwind end of every element.

Related theoretical results in the literature including superconvergence results and error estimates of the DG methods for ODEs are given in [33,27,26,32,30]. In particular, Lesaint and Raviart [33] studied the numerical solution of the initial value problem (1.1) by a DG method. Their scheme is equivalent to some implicit Runge–Kutta method, strongly A-stable one-step method. Delfour et al. [27] analyzed a class of Galerkin methods derived from discontinuous piecewise polynomial spaces. These schemes generalize the method proposed by Lesaint and Raviart [33]. In their DG method, the approximated solution at t_j, a point of discontinuity in the approximating polynomial u_h, is taken as an average across the jump: $\alpha_j u_h(t_j^-) + (1-\alpha_j)u_h(t_j^+)$. The cases $\alpha_j = 0, 0.5, 1$ correspond, respectively, to Euler's explicit, improved, and implicit schemes. Later, Delfour and Dubeau [26] studied the approximation of the solution of the nonlinear ODEs by discontinuous piecewise polynomials. They introduced a more general theory of one-step (such as implicit Runge–Kutta and Crank–Nicholson schemes), hybrid and multistep methods (such as Adams–Bashforth and Adams–Moulton schemes). Also, we mention the work of Johnson [32] in which *a priori* error estimates for a class of implicit one-step methods generated by the DG time discretization are proven. Estep [30] analyzed a finite element method for the integration of IVPs in ODEs. The author obtained quasi-optimal *a priori* and *a posterior* error bounds. They used these results to construct a rigorous and robust theory of global error control. The author also derived an asymptotic error estimate that is used in a discussion of the behavior of the error. Recently, Deng and Xiong [28] introduced and analyzed a DG finite element method with interpolated coefficients for an IVP of nonlinear ODE. They used the finite element projection for an auxiliary linear problem as comparison function and proved an optimal superconvergence results. Subsequently, Adjerid and Baccouch [4,11,10] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid et al. [6]. They proved that, for smooth solutions, these *a posteriori* error estimates at a fixed time t converge to the true spatial errors in the L^2-norm under