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Block extrapolation methods with applications

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ABSTRACT

In the present paper we introduce new block extrapolation methods as generalizations of the well known vector extrapolation methods. We give expressions of the obtained approximations via the Schur complement and also propose an efficient implementation of these methods. Applications to linearly generated sequences are given and extensions to nonlinear problems are also given. Applications of the proposed block extrapolation methods to some nonlinear matrix equations are considered and some numerical examples are given.

1. Introduction

Vector extrapolation methods are generalizations, to the vector case, of the well known Aitken Δ^2 process [2,8]. These vector sequence transformations methods are the minimal polynomial extrapolation (MPE) method of [13], the reduced rank extrapolation (RRE) [14,20] and the modified minimal polynomial extrapolation (MMPE) [6,21]. Analysis and computational procedures of these vector extrapolation methods could be found in [8,16,17,23,22]. A second class of vector sequence transformations contains the topological ϵ-algorithm (TEA) [8,6] and the vector ϵ-algorithms (VEA) [24]. Some matrix generalizations of these methods are given in [9,11,19,15].

In the present paper, we introduce block versions of these vector extrapolation methods. The use of the Schur complement simplifies the presentation of the proposed methods and could be used to develop algorithms for their implementation.

The paper is organized as follows. Section 2 recalls the classical vector extrapolation methods. In Section 3, we introduce the block versions of the vector polynomial extrapolation methods namely the block reduced rank extrapolation (BRRE), the block minimal polynomial extrapolation (BMPE) and the block modified minimal polynomial extrapolation (BMMPE). We also give a block version of the block topological ϵ-transformation. In Section 4, we give an efficient implementation of the block MMPE method. In Section 5 we apply these new block methods to linearly generated sequences for solving linear systems with multiple right-hand sides and show in this case that the approximations produced by these extrapolation methods are the same as those produced by well known block Krylov subspace methods. In Section 6, we give an extension to nonlinear matrix equations. Finally, some numerical tests are proposed and applications to nonlinear matrix equations are presented in Section 7.

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Notations. The Frobenius norm of a matrix X is defined as $\| X \|_F^2 = trace(X^T X)$ where $trace$ denotes the trace of the square matrix $X^T X$. If A is an $N \times N$ matrix and V is an $N \times s$ matrix, then the block Krylov subspace generated by the pair (A, V) is the subspace of $\mathbb{R}^N$ generated by the columns of $V, AV, \ldots, A^{m-1}V$ and denoted by

$$\mathscr{K}_m(A, V) = Range([V, AV, \ldots, A^{m-1}V]).$$

If M is the matrix partitioned as $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then the Schur complement of D in M, where D is square and nonsingular, is given by (see [10,25])

$$(M/D) = A - BD^{-1}C.$$

If the matrix A is nonsingular, we define the Schur complement of A in M as

$$(M/A) = D - CA^{-1}B.$$

We notice that in this case, M is nonsingular iff (M/A) is nonsingular.

2. Vector extrapolation methods

In this section, we review some well known polynomial vector extrapolation methods. Let (s_n) be a sequence of vectors in $\mathbb{R}^N$, then we define the new transformed sequence $s_k^{(n)}$ as follows

$$s_k^{(n)} = s_n + \sum_{i=1}^{k} a_i^{(n)} g_i(n), \tag{1}$$

where $g_i(n)$, $i = 1, \ldots, k$ is a given auxiliary sequence of $\mathbb{R}^N$ and related to the sequence (s_n). The scalar coefficients are obtained from the orthogonality relation

$$(y_j^{(n)})^T \widetilde{R}(s_k^{(n)}) = 0, \; j = 1, \ldots, k, \tag{2}$$

where $y_1^{(n)}, \ldots, y_k^{(n)}$ are some chosen vectors depending or not on the sequence (s_n), and

$$\widetilde{R}(s_k^{(n)}) = \Delta s_n + \sum_{i=1}^{k} a_i^{(n)} \Delta g_i(n). \tag{3}$$

The forward difference operator Δ acts on the index n, $(\Delta s_n = s_{n+1} - s_n)$. Different choices of the sequences $(g_i(n))_n$, $i = 1, \ldots, k$ and the vectors $y_j^{(n)}$, $j = 1, \ldots, k$ give the well known vector extrapolation methods:

- If $g_i(n) = \Delta s_{n+i-1}$ and $y_j^{(n)} = \Delta^2 s_{n+j-1}$ we obtain the RRE method.
- If $g_i(n) = \Delta s_{n+i-1}$ and $y_j^{(n)} = \Delta s_{n+j-1}$ then we get the MPE method.
- If the vectors $y_j(n) = y_j$, $j = 1, \ldots, n$ are fixed in $\mathbb{R}^N$ then the obtained method is MMPE.

In a matrix form, the approximation $s_k^{(n)}$ is given as follows

$$s_k^{(n)} = s_n - G_{k,n}(Y_{k,n}^T \Delta G_{k,n})^{-1} Y_{k,n}^T \Delta s_n, \tag{4}$$

where $G_{k,n} = [g_1^{(n)}, \ldots, g_k^{(n)}]$, $Y_{k,n} = [y_1^{(n)}, \ldots, y_k^{(n)}]$ and Δ acts on the index n. More developments on vector extrapolation methods are found in the papers [8,16–18,23,22]. In the next section, we define generalizations of these vector transformations to the matrix case.

3. The block extrapolation methods

3.1. The polynomial block extrapolation methods

Let $(S_n)_n$ be a sequence of matrices of $\mathbb{R}^{N \times s}$ and consider the sequence $(S_k^{(n)})_n$, $k = 1, 2, \ldots$ defined as follows

$$S_k^{(n)} = S_n + \sum_{i=1}^{k} G_i(n) A_i^{(n)}, \; n \geq 0 \tag{5}$$