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A Newton type linearization based two grid method for coupling fluid flow with porous media flow

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ABSTRACT

In this paper, we propose a two-grid finite element method for solving the mixed Navier–Stokes/Darcy model with the Beavers–Joseph–Saffman interface condition. After solving a coupled nonlinear problem on a coarse grid, we sequentially solve decoupled and linearized subproblems on a fine grid and then correct the solution on the same grid. Compared with the existing work on the two-grid methods for the coupled model, our two-grid method allows a much higher order scaling between the coarse grid size H and the fine grid size h. Specifically, if a k-th order discretization is applied, by using $h = H^{\frac{2k+1}{k}}$ for $k = 1, 2$ and $h = H^{\frac{k+1}{k-1}}$ for $k \geq 3$, the final step two-grid solution errors in the energy norm are still optimal. Numerical experiments are also given to confirm the theoretical analysis.

1. Introduction

In recent years, there are increasing interests in the coupling of fluid flow with porous media flow. Applications include the environmental engineering problem of groundwater contamination through rivers and the geoscience problem of surface flows filtrating in vuggy porous media. The most widely used models are based on Stokes/Darcy equations or Navier–Stokes/Darcy equations plus the so-called Beavers–Joesph–Saffmann or Beavers–Joesph interface conditions. These models are actually macro-scale sharp interface models for describing the interactions of fluid flow with porous media flow. Most existing works focus on solving the linear cases of the coupled models, see [3,6,9,11,13,17–19,23,29,30,32,35,36] for example. In this work, we aim at solving the coupled nonlinear Navier–Stokes/Darcy model. To deal with the difficulties caused by the coupling of different submodels and the nonlinearity [10,12,14,15,21,22], we propose a new two-grid algorithm based on the Newton type linearization [10,16,27]. The proposed two-grid algorithm consists of three steps: Firstly, the coupled model is solved on a coarse grid by using a nonlinear iterative algorithm; Secondly, on a fine grid level,

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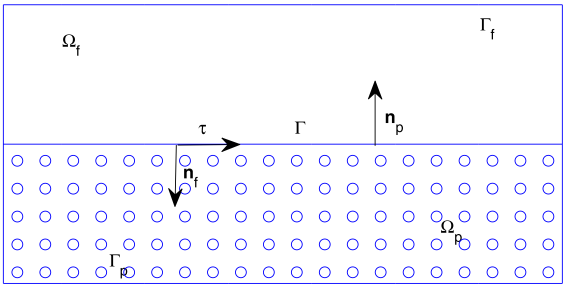

Fig. 1. A global domain Ω consisting of a fluid region Ω_f and a porous media region Ω_p separated by an interface Γ.

the Darcy problem is solved with the boundary condition at the interface provided by the coarse grid solution and then a Newton type linearization technique is adopted for the Navier–Stokes model; Thirdly, on the same fine grid triangulation, by using the newly obtained solutions, correction steps are sequentially applied to solve the Darcy model and the linearized Navier–Stokes model.

Let us review some existing work, in particular, the related two-grid methods for decoupling the coupled fluid/porous media flow models. For the mixed Stokes/Darcy model, Mu and Xu in [30] propose a two-grid method in which the coarse grid solution is used to supplement the boundary conditions at the interface for both of the two subproblems. If the first order discretization is applied, the scaling between the coarse meshsize and the fine meshsize needs to be taken as $h = H^{3/2}$ so that the two-grid solution errors for $\nabla\mathbf{u}$ and p are optimal. Recently, by firstly solving the Darcy problem on the fine grid and using the newly obtained Darcy's solution to provide the boundary condition for the Stokes model, a new two-grid method is proposed in [35,36]. It is shown that if $h = H^2$, the two-grid solution errors in the energy norm are optimal for all variables. This sequential solving technique can be naturally generalized to the nonlinear case [10,14,15,22,37], which actually leads to the first two steps of our two-grid algorithm. However, even for the two-grid method based on the first two steps, the L^2 error analysis needs to be further explored. Moreover, we note that the existing research is mainly based on the first order discretization. Different from the previous work, we apply a general k-th order discretization. Furthermore, we observe that the scaling between the two meshsizes can be further improved if the correction step is applied. Based on a detailed error analysis in both the L^2 norm (see Theorem 4.1) and the energy norm (see Lemma 4.1 and Theorem 4.2), we show that by using $h = H^{\frac{2k+1}{k}}$ for $k = 1, 2$ and $h = H^{\frac{k+1}{k-1}}$ for $k \geq 3$, the final step two-grid solution errors in the energy norm are still optimal. Besides of the above mentioned advantages, our algorithm actually solves the same two linear subproblems with only different loads in the last two steps of the algorithm.

The rest of the paper is organized as follows. In Section 2, we introduce the coupled Navier–Stokes/Darcy systems. In Section 3, the finite element approximations and the new two-grid algorithm are presented. In Section 4, we present the theoretical analysis. Numerical experiments are given in Section 5 to verify the theoretical predictions. Finally, the corresponding two-grid algorithm for the linear Stokes/Darcy case is presented in the appendix.

2. Coupled Navier–Stokes and Darcy systems

Let $\Omega \subset R^d$ be a domain consisting of a fluid region Ω_f and a porous media region Ω_p separated by an interface Γ, as shown in Fig. 1, where $d = 2$ or 3, $\Omega = \Omega_f \bigcup \Omega_p$ and $\Gamma = \overline{\Omega}_f \bigcap \overline{\Omega}_p$. Let $\mathbf{n}_f$ and $\mathbf{n}_p$ denote the unit outward normal directions on $\partial\Omega_f$ and $\partial\Omega_p$. The interface Γ is assumed to be smooth enough as in [21].

The fluid flow in Ω_f is governed by the steady state Navier–Stokes equations:

$$\begin{cases} -\nu\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}_f & \text{in} \quad \Omega_f, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega_f, \end{cases} \tag{2.1}$$

where ρ is the density of the fluid flow, $\mathbf{u}$ is the velocity vector, p is the pressure, $\mathbf{f}_f$ is the external force, $\nu > 0$ is the viscosity coefficient.

The flow motion in the porous media region Ω_p is modeled by Darcy's law, namely,

$$\begin{cases} \mathbf{q} = -\mathbf{K}\nabla\phi & \text{in} \quad \Omega_p, \\ \nabla\cdot\mathbf{q} = f_p & \text{in} \quad \Omega_p. \end{cases} \tag{2.2}$$

Here, $\mathbf{q} = n\mathbf{u}_p$ is the discharge vector with $\mathbf{u}_p$ being the velocity and n being the volumetric porosity, $\mathbf{K} = \frac{\epsilon^2}{\nu}\mathbf{I}$ is the hydraulic conductivity tensor with ϵ being the characteristic length of the porous media, f_p is a source term due to injection or pump, and

$$\phi = z + \frac{p_p}{\rho g}$$