Applied Numerical Mathematics 96 (2015) 108–117

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Uniformly convergent difference schemes for a singularly perturbed third order boundary value problem

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ARTICLE INFO

Article history:
Received 10 April 2014
Received in revised form 15 April 2015
Accepted 2 June 2015
Available online 6 June 2015

Keywords:
Singular perturbation
Boundary value problem
Layer-adapted meshes
Finite difference method

ABSTRACT

In this paper we consider a numerical approximation of a third order singularly perturbed boundary value problem by an upwind finite difference scheme on a Shishkin mesh. The behavior of the solution, and the stability of the continuous problem are discussed. The proof of the uniform convergence of the proposed numerical method is based on the strongly uniform stability and a weak consistency property of the discrete problem. Numerical experiments verify our theoretical results.

1. Introduction

We consider the boundary value problem

$$Lu := \varepsilon u'''(x) + a(x)u''(x) + a_1(x)u'(x) + a_0(x)u = f(x), \quad \text{for} \quad x \in (0,1)$$
$$u(0) = u'(0) = 0, \quad u(1) = 0, \tag{1.1}$$

where ε is a small positive parameter, $a(x) > \alpha > 0$, and the coefficients are infinitely differentiable. The second order or the fourth order singularly perturbed problems are the ones most usually studied. However, there are several applications of third order problems as well, see for instance [4] or [5]. Although the applications mentioned in [4] and [5] have a structure different from (1.1), we find it useful and instructive to study (1.1) first.

Our aim is to develop a uniformly convergent (or robust) numerical method of order p which solves (1.1). Recall, a numerical method with N degrees of freedom is uniformly convergent of order p if

$$\|u - u^N\| \le C\, N^{-p}, \tag{1.2}$$

where $\|\cdot\|$ is an appropriate norm and the constant $C > 0$ is independent of ε and N.

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[1] The work of Lj. Teofanov was supported by the Ministry of Science and Technological Development of the Republic of Serbia under Grant 174030. The work of Z. Uzelac was supported by the Ministry of Science and Technological Development of the Republic of Serbia under Grants 174030 and III44006.

http://dx.doi.org/10.1016/j.apnum.2015.06.002
0168-9274/

In [14] the authors examined the boundary value problem

$$Lu := \varepsilon u'''(x) + a(x)u''(x) + a_1(x)u'(x) + a_0(x)u = f(x), \quad \text{for} \quad x \in (0,1)$$
$$u(0) = u'(0) = 0, \quad u'(1) = 0. \tag{1.3}$$

This problem is easier than problem (1.1) because the boundary values allow the transformation of (1.3) into a weakly coupled system of a first order equation and a second order equation.

Gartland [3] studied compact difference schemes (for even more general problems) on a locally uniform exponentially graded mesh. But because of the requirement that in some transition region the scheme is exact for some polynomials and the function $\exp(-\frac{1}{\varepsilon}\int_0^x a)$, those schemes are rather complicated. It is also possible, for example, to transform the problem to a first order system and then to apply techniques from [2]. However, the results proved in [2] are not strongly uniform in the sense of (1.2).

Our aim is to study a standard upwind difference scheme on the simplest layer adapted mesh – a piecewise equidistant Shishkin mesh. For the second order boundary value problems with exponential boundary layers it is known that for simple upwinding on a Shishkin mesh we have the following estimate

$$|u(x_i) - u_i| \lesssim N^{-1} \ln N \tag{1.4}$$

($a \lesssim b$ means: there exists a positive constant C independent of ε and N such that $a \le Cb$). The above result was published in [1, Remark 5] for the first time. For the same second order problem, in [7] the authors proved $|u(x_i) - u_i| \lesssim N^{-1}(\ln N)^2$ using the discrete maximum principle and majorizing functions. The improved estimate (1.4) can be obtained with a more refined barrier function, see [11, Ch. I.2.4] or [12]. However, the problem (1.1) fails to satisfy the maximum principle. Therefore, we follow the idea of Andreev and Savin [1] for the second order problems, i.e. we use $L_\infty - L_1$ stability and measure the consistency error in the discrete L_1 norm. Based on this, we shall prove that for (1.1) there is an upwind scheme on a Shishkin mesh which also gives (1.4).

Regarding the finite elements, in [13] the authors study only the even order problems on Shishkin meshes. However, we do not know of any results for odd order singularly perturbed problems.

2. The continuous problem

In [10], O'Malley describes in detail asymptotic expansions for higher order equations. Let us repeat the basic facts for the problem (1.1). The cancellation law tells us that the so-called reduced problem (when $\varepsilon = 0$) is defined by

$$a(x)z''(x) + a_1(x)z'(x) + a_0(x)z = f(x) \quad \text{in } (0,1), \qquad z(0) = z(1) = 0. \tag{2.1}$$

Recall, the cancellation law specifies the boundary condition which should be discarded from (1.1) when setting the reduced problem (see [11, p. 35] for more details). The following analysis is based on Theorem 2 [10, Ch. 3].

We assume that (2.1) is well defined and that z is uniquely determined. Then, for $\varepsilon \le \varepsilon_0$ the problem (1.1) has a unique solution of the form

$$u(x) = G(x,\varepsilon) + \varepsilon \tilde{G}(x,\varepsilon) \exp\left(-\frac{1}{\varepsilon}\int_0^x a(t)dt\right), \tag{2.2}$$

where G and $\tilde{G}$ have asymptotic power series expansions in ε, and $G(x,0) = z(x)$, $x \in [0,1]$. Moreover, the representation (2.2) can be repeatedly differentiated. Then it follows

$$|u^{(k)}(x)| \lesssim (1 + \varepsilon^{-k+1}\exp(-\alpha x/\varepsilon)) \tag{2.3}$$

or, equivalently, the existence of an S-decomposition $u = S + E$ with

$$|S^{(k)}(x)| \lesssim 1, \quad |E^{(k)}(x)| \lesssim \varepsilon^{-k+1}\exp(-\alpha x/\varepsilon). \tag{2.4}$$

We call the problem (1.1) strongly uniformly stable if the estimate

$$\|u\|_\varepsilon \lesssim \{\|Lu\|_1 + |u(0)| + |u'(0)| + |u(1)|\} \tag{2.5}$$

is satisfied for all sufficiently smooth functions u, where $\|v\|_\varepsilon := \max\{\|v\|_\infty, \|v'\|_\infty, \varepsilon\|v''\|_\infty\}$. Here $\|\cdot\|_1$ denotes L_1 norm.

From Theorem 1.2 [3] we have the following assertion:

Let $\{\phi_\nu\}_{\nu=1,2,3}$ be a fundamental system of solutions of L that satisfies

$$\|\phi_\nu\|_\varepsilon \le C, \tag{2.6}$$

and is such that the matrix

$$\begin{bmatrix} \phi_1(0) & \phi_1'(0) & \phi_1(1) \\ \phi_2(0) & \phi_2'(0) & \phi_2(1) \\ \phi_3(0) & \phi_3'(0) & \phi_3(1) \end{bmatrix} \text{ has an inverse whose norm is uniformly bounded.} \tag{2.7}$$

Then, the problem (1.1) is strongly uniformly stable.