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Penalization of Robin boundary conditions

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ABSTRACT

This paper is devoted to the mathematical analysis of a method based on fictitious domain approach. Boundary conditions of Robin type (also known as Fourier boundary conditions) are enforced using a penalization method. A complete description of the method and a full analysis are provided for univariate elliptic and parabolic problems using finite difference approximation. Numerical evidence of the predicted estimations is provided as well as numerical results for a nonlinear problem and a first extension of the method in the bivariate situation is proposed.

1. Introduction

This paper is devoted to the definition and analysis of a finite difference fictitious domain method where the boundary conditions of Robin type are enforced using a penalization method. As for all fictitious domain methods, the initial problem with solution u and raised on a domain ω is solved using a larger but simpler domain $\Omega \supset \omega$. In penalization methods, the new problem defined on Ω is parameterized by η defined in such a way that its solution u_η restricted to ω converges, when η goes to zero, towards u the solution of the initial problem.

Penalization has been introduced by Arquis and Caltagirone [5] in the 80s and analyzed by Angot et al. [3] for incompressible Navier–Stokes equations. Various theoretical results have been established for Dirichlet boundary conditions, in the case of parabolic [18] or hyperbolic [22,13,14,10] equations. The main advantage of penalization methods stands in allowing the numerical resolution of the problem in an obstacle-free simple domain. There, the use of a Cartesian mesh is possible and different numerical methods including pseudo-spectral methods [18], finite differences/volumes [3,20,22] and wavelets [26,27,19] can be implemented efficiently to approximate the solution.

This paper can be considered as a continuation of [16] where Dirichlet boundary conditions were considered. The scope of that paper was the modeling of the plasma–wall interaction in the region of a tokamak called Scrape-Off Layer (SOL), where magnetic field lines are open and intercept solid obstacles. A volume penalization method has been proposed to take into account the boundary conditions associated with a hyperbolic model system for the density and the momentum of the plasma through a magnetic field line in the SOL region. A modification of this method has been given in [4] to eliminate the presence of an artificial boundary layer. Enriching the physical model of the edge plasma requires adding the evolution of ionic and electronic temperatures. These quantities are modeled by parabolic equations with boundary conditions of Neumann or Robin types. This extension of the physical model is the starting point and the initial motivation of the present

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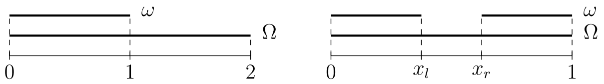

Fig. 1. Left: geometrical domain for theoretical analysis. Right: geometrical domain used for applications. In both cases ω is the initial domain, Ω is the fictitious domain.

work, since few penalization methods for this type of conditions exist in the literature, see for instance [2,24,25,17]. As in [16], univariate problems are first considered. For the theoretical analysis of the penalization, the initial problem is raised on $\omega =]0,1[$ and the fictitious domain is defined as $\Omega =]0,2[$. For the numerical approximation, a more realistic situation inspired by tokamak geometry is considered: the domain ω is $]0, x_l[\cup]x_r, 1[$ and $\Omega =]0,1[$ with $0 < x_l < x_r < 1$.

The paper is organized as follows: Section 2 is devoted to elliptic equations. The penalized problem is introduced and existence, uniqueness and convergence of the solution in regard to penalization parameter are established. Numerical approximation of the problem using finite differences is defined and analyzed in Section 2.6. Convergence and stability analysis are provided and illustrated by various numerical tests. Parabolic equations are considered in Section 3 where a time discretization followed by space discretization are used. The analysis is performed and illustrated by various simulations. Section 4 is devoted to numerical tests for a nonlinear problem describing the temperature in the plasma of a tokamak close to an obstacle. A first application of the method to the bivariate situation is also presented.

2. Elliptic equation

2.1. Definition of the problem

An archetype of elliptic partial differential equations with Robin boundary conditions is the so called reaction–diffusion equation

$$\begin{cases} -\Delta u + u = f & \text{in } \omega \\ \dfrac{\partial u}{\partial n} + \alpha u = g & \text{on } \partial\omega, \end{cases}$$

where ω is a given smooth bounded open set in $\mathbb{R}^D$, n is the outward-pointing unit normal vector on the boundary $\partial\omega$, $f \in L^2(\omega)$, $g \in L^2(\partial\omega)$ and $\alpha \geq 0$ are given. The case $\alpha = 0$ corresponds to Neumann boundary conditions. The Lax–Milgram theorem [1,11] provides the existence and uniqueness of the solution $u \in H^1(\omega)$ of its weak formulation. In the particular case $D = 1$ and $\omega =]0,1[$, the problem reads

$$\begin{cases} -u'' + u = f & \text{in }]0,1[\\ -u'(0) + \alpha u(0) = g(0), \ u'(1) + \alpha u(1) = g(1). \end{cases} \tag{2.1}$$

The unique solution $u \in H^1(]0,1[)$ of the weak formulation of (2.1) is in fact in $H^2(]0,1[)$ taking into account that $u'' = u - f \in L^2(]0,1[)$. It follows using the Sobolev embedding $H^2(]0,1[) \hookrightarrow C^1([0,1])$ that $u \in C^1([0,1])$ and, if $f \in C^0([0,1])$ that $u \in C^2([0,1])$.

As presented on Fig. 1 left, $\Omega =]0,2[$ is used to define a penalized problem associated with (2.1). Denoting χ the characteristic function of $\Omega \setminus \bar{\omega} =]1,2[$ and introducing a real parameter $\eta > 0$, a new problem raised on Ω with Robin boundary conditions at the boundary points reads

$$\begin{cases} -u_\eta'' + u_\eta + \dfrac{\chi}{\eta}\big(u_\eta' + \alpha u_\eta - g(1)\big) = (1-\chi) f & \text{in }]0,2[\\ -u_\eta'(0) + \alpha u_\eta(0) = g(0), \ u_\eta'(2) + \alpha u_\eta(2) = g(1). \end{cases} \tag{2.2}$$

When $x \in]0,1[$, $\chi(x) = 0$ and we recover the equation of (2.1) with the Robin condition at $x = 0$ unchanged. When $x \in]1,2[$, $\chi(x) = 1$, and using a small value of η implies that the leading term in the first equation of (2.2) is $\frac{1}{\eta}(u_\eta' + \alpha u_\eta - g(1))$ which imposes $u_\eta' + \alpha u_\eta - g(1) \approx 0$. We therefore recover, at least formally, the boundary conditions of (2.1).

Remark 2.1. Boundary conditions at $x = 2$ have been set to Robin type but different conditions could be enforced at this point.

In the following sections, we prove that this penalized problem (2.2) is well-posed in its weak form: