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Timestepping schemes for the 3d Navier–Stokes equations

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ABSTRACT

It is well known that the (exact) solutions of the 3d Navier–Stokes equations remain bounded for all time if the initial data and the forcing are sufficiently small relative to the viscosity. They also remain bounded for a finite time for arbitrary initial data in L^2. In this article, we consider two temporal discretisations (semi-implicit and fully implicit) of the 3d Navier–Stokes equations in a periodic domain and prove that their solutions remain uniformly bounded in H^1 subject to essentially the same respective smallness conditions as the continuous system (on initial data and forcing or on the time of existence) provided the time step is small.

1. Introduction

Much work has been done on the stability and convergence of various timestepping schemes for the Navier–Stokes equations in two space dimensions (2d NSE). The long time stability in H^1 of the implicit Euler scheme for the 2d NSE has been treated in, e.g., [2,6,4,8], and more recently extended to higher-order schemes in [9,3]. Once the numerical solutions are shown to be bounded in suitable space, either on a limited interval of time or for all time, convergence can usually be established using standard techniques (cf., e.g., [5]).

In three dimensions, the existence of a solution bounded in $L^\infty(\mathbb{R}_+, H^1)$ is not known; we only know the existence of a globally bounded solution if the data are small and of a locally bounded solution for data of arbitrary size; for more background on the NSE, see e.g. [1,7]. Hence, the extension of the numerical stability results from 2d or 3d is not straightforward. We conduct it here in the two cases for which the existence of a strong solution is known, namely, as we said, a globally bounded solution associated with small data or a locally bounded solution associated with arbitrary data.

In this article we consider temporal discretisations of the 3d NSE using a semi-implicit Euler scheme and the fully implicit Euler scheme; see (2.1) and (3.1) below. For both of these schemes, we establish the uniform boundedness in H^1 for the discrete solutions for interval of times corresponding to the existence of solutions. As in the earlier works cited above, we do not consider time and space discretisations, giving the advantage that our results will be free of Courant–Friedrichs–Lewy-type constraints, although some smallness of the timestep is required.

We consider the Navier–Stokes equations in $\Omega = (0, 2\pi)^3$ with periodic boundary conditions,

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f,$$

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$$\nabla\cdot u=0, \tag{1.1}$$

plus the initial data $u(0)=u_0$. With no loss of generality, we assume that $\nabla\cdot f=0$, and that the integrals of f and u_0 vanish over Ω. The last assumption implies that $u=u(t)$, whenever it is well-defined for $t\ge 0$, also has vanishing integral over Ω, giving us the Poincaré inequality

$$|u|_{L^2}^2\le c_0(\Omega)|\nabla u|_{L^2}^2. \tag{1.2}$$

For notational convenience, we redefine c_0 to give also the bound (for u periodic with average 0):

$$|\nabla u|_{L^2}^2\le c_0|\Delta u|_{L^2}^2. \tag{1.3}$$

In order to facilitate comparison with the numerical solutions, in the rest of this section we briefly review the boundedness of solutions of the 3d NSE, both in L^2 and in H^1 for the two cases (small data, large time and short time for arbitrary data).

Multiplying (1.1) by u in $L^2(\Omega)$, integrating by parts and using the fact that $\big((u\cdot\nabla)u,u\big)=0$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+\nu|\nabla u|^2=(f,u). \tag{1.4}$$

Here and henceforth, unadorned norm $|\cdot|$ and inner product $(\cdot,\cdot)$ are taken to be L^2. Bounding the rhs by the Cauchy–Schwarz inequality and using the Poincaré inequality, (1.4) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+\frac{\nu}{c_0}|u|^2\le\frac{1}{\nu}|f|_{L^\infty(H^{-1})}^2, \tag{1.5}$$

where $|f|_{L^\infty(H^{-1})}:=\sup_{t\ge 0}|f(t)|_{H^{-1}}$. Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2\exp\left(\frac{\nu}{c_0}t\right)\right)\le\exp\left(\frac{\nu}{c_0}t\right)\frac{1}{\nu}|f|_{L^\infty(H^{-1})}^2. \tag{1.6}$$

Integrating from 0 to t (here we change the dummy variable of integration to s), we obtain

$$|u(t)|^2\exp\left(\frac{\nu}{c_0}t\right)\le|u(0)|^2+\frac{c_0}{\nu^2}|f|_{L^\infty(H^{-1})}^2\left(\exp\left(\frac{\nu}{c_0}t\right)-1\right). \tag{1.7}$$

We then find the uniform L^2 bound valid for all $t\ge 0$:

$$|u(t)|^2\le|u(0)|^2+(c_0/\nu^2)|f|_{L^\infty(H^{-1})}^2=:K_0(u_0,f;\nu,\Omega). \tag{1.8}$$

1.1. H^1 estimate for small data

Now multiplying (1.1) by $-\Delta u$ in $L^2(\Omega)$ and integrating by parts, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2+\nu|\Delta u|^2=\big((u\cdot\nabla)u,\Delta u\big)-(f,\Delta u). \tag{1.9}$$

Bounding the nonlinear term using the Sobolev inequality $|u|_{L^6}\le c|\nabla u|_{L^2}$, which is specific to dimension three,

$$\big|\big((u\cdot\nabla)u,\Delta u\big)\big|\le|u|_{L^3}|\nabla u|_{L^6}|\Delta u|_{L^2}\le\frac{c_1}{2}|u|_{L^3}|\Delta u|_{L^2}^2, \tag{1.10}$$

and the forcing term in the obvious fashion

$$(f,\Delta u)\le\frac{1}{\nu}|f|^2+\frac{\nu}{4}|\Delta u|^2, \tag{1.11}$$

we then arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u|^2+(3\nu/2-c_1|u|_{L^3})|\Delta u|^2\le\frac{2}{\nu}|f|^2. \tag{1.12}$$

Assuming that

$$|u_0|_{L^3}\le\frac{\nu}{4c_1}, \tag{1.13}$$

we find that on some interval of time $(0,T)$

$$|u|_{L^3}\le\nu/(2c_1). \tag{1.14}$$