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A note on the stability of cut cells and cell merging

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ABSTRACT

Embedded boundary meshes may have cut cells of arbitrarily small volume which can lead to stability problems in finite volume computations with explicit time stepping. We show that time step constraints are not as strict as often believed. We prove this in one dimension for linear advection and the first order upwind scheme. Numerical examples in two dimensions demonstrate that this carries over to more complicated situations. This analysis sheds light on the choice of time step when using cell merging to stabilize the arbitrarily small cells that arise in embedded boundary schemes.

1. Introduction

Cell merging is a commonly used technique for cut cells in embedded boundary grids. It creates a cell with a larger volume, thus allowing an explicit difference scheme to use a larger time step and still remain stable on the small cut cells. A cell is usually merged with its neighbor in the direction most normal to the boundary [5,2]. Typically, cut cells whose volume is less than 0.5 of the regular cell volume are merged. Then a time step that is somewhat reduced to account for these smaller cell volumes is used for simulations. Some papers do not report the actual timestep used on their cut cell meshes [8]; others reduce the time step proportionately with the volume fraction of the cut cell [4]. The stability of cell merging is not well understood.

This note gives a theoretical analysis and some numerical examples of the stability of cut cells, to shed some light on the stability of cell merging. We use a one dimensional model problem and apply GKS stability theory. In two dimensions we show computations that suggest that the conclusions of this analysis also apply in more complicated situations, although the results are not as clear cut. Related work was done in [6] but focused on discontinuous Galerkin methods. This note is hopefully just a beginning, and will serve to stimulate more research in this area.

In more detail, we prove that one can take a full time step Δt if the cut cells at the boundary are at least half the regular cell volume for the first order upwind finite volume scheme. For the second order MUSCL scheme, the stability limit varies, depending on the details of the gradient computation. We will show that using central differences for the gradient has better stability properties than downwind gradients. This latter is equivalent to the Lax Wendroff scheme, whose stability limit is slightly worse that the first order case, suggesting that a cell's volume after merging should be somewhat larger than half. We also present numerical experiments for a linear and nonlinear problem in two dimensions on an embedded boundary grid to investigate the extent to which the one dimensional results carry over.

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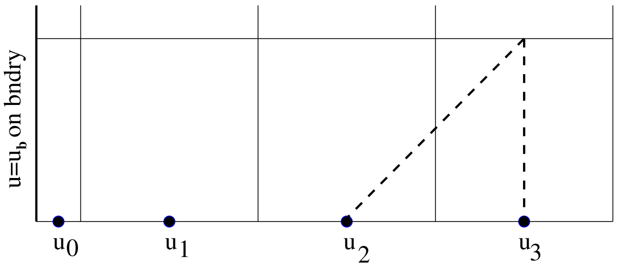

Fig. 1. Uniform Cartesian mesh with one small cell at an inflow boundary. The dotted line indicates the stencil for the first order upwind scheme.

2. Cut cell model problem in one space dimension

We use the following model problem sketched in Fig. 1 to show stability for linear advection $u_t + au_x = 0$. The cut cell is at the boundary, to mimic the situation in higher dimensions. This is in contrast to some of our previous work where we put the cut cell in the middle of the grid [1]. The main tool we use is the theory of Gustafsson, Kreiss and Sundström (GKS) [3]. Consider $a > 0$ so that the cut cell is at an inflow boundary, where $u(x = 0) = u_b$. We look at both the first and second order case.

2.1. First order case

The explicit first order upwind scheme for this problem is

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n), \qquad j \geq 1 \tag{1}$$

where the notation is illustrated in Fig. 1. The von Neumann stability condition for the first order upwind scheme is $\lambda \leq 1$, where the CFL number $\lambda = a\Delta t/h$. The small cell is the first cell, u_0 in the figure, with mesh width αh, $0 < \alpha \leq 1$. The conservative finite volume update for this cell is

$$u_0^{n+1} = u_0^n - \frac{a\Delta t}{\alpha h}(u_0^n - u_b^n) \tag{2}$$

For the stability analysis, it suffices to take $u_b = 0$.

GKS theory looks for solutions of the form

$$u_j^n = z^n v_j, \tag{3}$$

with the magnitude of the complex eigenvalue parameter $|z|$ determining stability. If $|z| < 1$ there is no growth in time; if $|z| > 1$ the growth is exponential. The borderline cases of $|z| = 1$ are more complicated and need further testing, but they are not an issue in this work since the numerical scheme is dissipative. (See [9] for a very readable presentation of GKS theory.) After substituting (3) into the difference scheme, one obtains a recurrence equation based on the finite difference scheme. If it has any l^2 solutions, this is an unstable solution v_j.

Substituting (3) into the small cell boundary condition (2) gives

$$(z-1)v_0 + \frac{a\Delta t}{\alpha h} v_0 = 0 \qquad \text{or} \tag{4}$$

$$(z - 1 + \frac{\lambda}{\alpha})\, v_0 = 0. \tag{5}$$

From (5) we see that either $z - 1 + \lambda/\alpha = 0$ or $v_0 = 0$. The first case gives $z = 1 - \lambda/\alpha$, so for $|z| \leq 1$ we must have $\lambda/\alpha \leq 2$, or alternatively we need $\alpha \geq \lambda/2$ to guarantee stability. To put it a different way, in (2) we can take a small cell with mesh width αh at least half the uniform h and retain stability for all $\lambda \leq 1$. This means that the CFL number in the small cell $\frac{a\Delta t}{\alpha h}$ can exceed one!

The stability of the small cell scheme (1) and (2) is illustrated numerically in Fig. 2. We take 1000 steps using $\lambda = 0.9$ on a mesh with 100 cells, with exact solution $u(x,t) = \cos(2\pi(x-t))$, so the boundary condition $u_b(t) = \cos(2\pi t)$ is advecting into the domain, rather than the zero inflow conditions for analysis. We take the small cell parameter $\alpha = 0.46$ so that α is slightly above the stable limit of $\lambda/2 = 0.45$. After 1000 steps the computed solution is plotted on the left, with the 2 norm of the error 3.47×10^{-2}. Everything looks stable, as expected. On the right is an experiment with $\alpha = 0.40$, just slightly below the stability limit. The error after 40 steps is 0.16 in the 2 norm and 1.18 in the maximum norm. The instability is clearly visible emanating from the boundary.

Note that since the coefficients of the update for u_0 in (2) are no longer positive, a monotone solution is not guaranteed. But it is stable using the GKS definition, which essentially allows bounded growth independent of Δt. In fact this boundary