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Modified fully discretized projection method for the incompressible Navier–Stokes equations

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ABSTRACT

The stability and convergence of a second-order fully discretized projection method for the incompressible Navier–Stokes equations is studied. In order to update the pressure field faster, modified fully discretized projection methods are proposed. It results in a nearly second-order method. This method sacrifices a little of accuracy, but it requires much less computations at each time step. It is very appropriate for actual computations. The comparison with other methods for the driven-cavity problem is presented.

1. Introduction

To solve the incompressible Navier–Stokes equations, the projection method (or fractional step method) was originally introduced and studied independently by [3,4] and [24,25]. It has had tremendous applications, see e.g., [26,14,28,6], and the references therein, for the theoretical and numerical aspects. Despite many advantages and extensive uses in the past by numerous researchers, the projection method has a few major drawbacks for numerical computations. In general, the original method is only first-order accurate in time. It also needs the supplementary boundary conditions for the intermediate level velocity field and the pressure, which are not supplied in the original equations.

In order to improve the accuracy, modified second-order projection methods were introduced. At least three types of higher order projection methods were proposed. Namely, a method via improved intermediate velocity boundary conditions [14,7,10], a method via pressure-correction [27,1,8,22] and a method via improved pressure boundary condition [18,13,21,2]. However, all these methods still need supplementary boundary conditions. For a survey of these methods we refer the reader to, e.g., [16,9].

A fully discretized projection method was studied in [11,12] on the staggered grid. The idea was originated in the block LU decomposition, see [19]. This led to a whole class of methods (first-order, second-order and even higher order methods). Depending on how the Navier–Stokes equations are discretized, it is possible to construct higher order methods. However, for all methods with order of accuracy higher than 2 we need to solve a linear system and an inverse system. The computations are very costly.

In this article, we investigate the stability and convergence of the second-order fully discretized projection method proposed in [11,12]. With the help of approximation of inverse matrix, we proposed the modified fully discretized projection methods. Those methods have the accuracies between one and two. In order to have the accuracy closer to two, more iterations are needed.

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This article is organized as follows. In Section 2, we recall the Navier–Stokes equations and the boundary conditions. The space discretization is listed in Section 3. The fully discretized projection method is shown in Section 4. In Section 5, we present fully discretized projection methods. In Section 6 stability and convergence of second-order fully discretized projection method is studied. The modified fully discretized projection methods are shown in Section 7. Section 8 contains the numerical simulations of the driven-cavity problem and comparison with other results. We discuss the future work in Section 9.

2. Incompressible Navier–Stokes equations

We will consider the non-dimensionalized unsteady incompressible Navier–Stokes equations in space dimension two or three on a given regular domain Ω, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \dfrac{1}{Re}\Delta u + \nabla p = f, \\ \nabla\cdot u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{2.1}$$

with the boundary conditions on $\partial\Omega$:

$$\eta u + (1-\eta)\frac{\partial u}{\partial \vec{n}} = 0.$$

Here Re is the Reynolds number; the parameter η has the limit values of 0 for the free-slip (no stress) condition (Neumann) and 1 for the no-slip condition (Dirichlet). In general, we will not specify η, but keep in mind that $0 \le \eta \le 1$.

Actually, the boundary conditions do not affect the scheme as far as stability and convergence are concerned, but they are of course important for the numerical stability conditions. For the stability and convergence, what we need is the condition that guarantees the existence of the solution for the original equations.

3. Space discretization

Two families of finite-dimensional Hilbert spaces X_h and V_h are given, which depend on a parameter $h \in R^d_+$ $(d = 2, 3)$. For finite differences, h is the mesh, i.e. $h = \{h_1, h_2\} = \{\Delta x, \Delta y\}$ in space dimension two and $h = \{h_1, h_2, h_3\} = \{\Delta x, \Delta y, \Delta z\}$ in space dimension three.

Two scalar products $((\cdot,\cdot))_h$ and $(\cdot,\cdot)_h$ with corresponding norms $||\cdot||_h$ and $|\cdot|_h$ are defined on each V_h. Since V_h is a finite-dimensional space the two norms $||\cdot||_h$ and $|\cdot|_h$ are equivalent. We assume that they are related as follows

$$|u_h|_h \le c_1 ||u_h||_h, \tag{3.1}$$

$$||u_h||_h \le S(h)|u_h|_h, \qquad \forall u_h \in V_h, \tag{3.2}$$

where c_1 is independent of h and $S(h)$ depends on h.

When convergence will be studied, we will be interested in the passage to the limit $h \to 0$. The spaces V_h with scalar product $((\cdot,\cdot))_h$ will approximate in some sense the space V, while the spaces V_h with scalar product $(\cdot,\cdot)_h$ will approximate the space H. We assume the following

$$S(h) \to \infty, \text{ as } h \to 0$$

A trilinear operator b_h is defined on $V_h \times V_h \times V_h$ as follows:

$$b_h(u_h, v_h, w_h) = ((u_h\cdot\nabla)v_h, w_h), \quad \forall u_h, v_h, w_h \in V_h,$$

and we have the properties:

$$|b_h(u_h, v_h, w_h)| \le c_2 |u_h|_h^{\frac12} ||u_h||_h^{\frac12} ||v_h||_h |w_h|_h^{\frac12} ||w_h||_h^{\frac12}, \quad \forall u_h, v_h, w_h \in V_h, \tag{3.3}$$

in space dimension two, and

$$|b_h(u_h, v_h, w_h)| \le c_2 |u_h|_h^{\frac14} ||u_h||_h^{\frac34} ||v_h||_h |w_h|_h^{\frac14} ||w_h||_h^{\frac34}, \quad \forall u_h, v_h, w_h \in V_h, \tag{3.4}$$

in space dimension three. The constant c_2 in (3.3) and (3.4) is independent of h.

We also assume the skewness property:

$$b_h(u_h, v_h, v_h) = 0, \quad \forall u_h, v_h \in V_h, \tag{3.5}$$

which implies