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# Some remarks on finite element approximation of multiple eigenvalues

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ABSTRACT

In this paper we investigate the behavior of the finite element approximation of multiple eigenvalues in presence of eigenfunctions with different smoothness. We start from a one-dimensional example presented in the Handbook of Numerical Analysis by Babuška and Osborn and extend it to higher order approximation and to two dimensions, confirming that the different regularities of the eigenfunctions are well seen in the numerical computations. Then we discuss a mixed formulation corresponding to the one-dimensional example. It turns out that the regularity properties of the eigenfunctions are not well separated in this particular example, since the estimates have to take into account both components of the solution.



## 1. Introduction

The finite element approximation of variationally posed eigenvalue problems has been the object of a wide investigation. Standard estimates (see [1], for instance) imply that in the case of compact and symmetric operators an eigenvalue of multiplicity $m$ is approximated by exactly $m$ discrete eigenvalues (counted according to their multiplicities) and that the order of convergence is related to the approximation properties of the corresponding continuous eigenspace. If a multiple eigenvalue $\lambda$ is related to eigenfunctions with different smoothness, then from the classical theory it follows that the speed of convergence is related to the approximation properties of the eigenfunction with lowest regularity. A sharper analysis, presented in [5] and reviewed in [3] for the situation of interest, shows that indeed the discrete eigenvalues approximating $\lambda$ inherit different regularities according to the various approximation properties of the continuous eigenfunctions. The analysis applies to standard Galerkin approximation of symmetric compact operators and is based on the Ritz approximation of the solution operator and on the monotonicity property coming from the characterization of the eigenvalues in terms of Rayleigh quotient.

In Section 2 we introduce our notation and recall the main results of the analysis of Knyazev and Osborn [5].

In Section 3 we elaborate on a one-dimensional example presented in [1], which has been cooked up in order to have double eigenvalues associated with one singular and one more regular eigenfunction. With piecewise linear elements we can reproduce the results of Babuška and Osborn [1]; here we extend the computation to quadratic elements: in this case it is possible to appreciate that neither eigenfunction associated with each double eigenvalues can be approximated with optimal convergence rate, even if one eigenfunction is more regular than the other. In any case, the numerical tests confirm the theoretical results.

In Section 4 we extend the one-dimensional example to two dimensions. It turns out that the corresponding multiple eigenvalues have now multiplicity equal to 4 and that only one of the four associated eigenfunctions is regular. The numerical results, computed with piecewise bilinear elements, are in good agreement with the theoretical results.

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Finally, in Section 5 we study a mixed formulation of the one-dimensional example. This investigation is appealing, since the theory of Knyazev and Osborn [5] does not extend trivially to this context (there is no monotonicity coming from the Rayleigh quotient). We derive an explicit formula for the error in the eigenvalues (see also [4] for a similar formula in a more standard situation) from which it is clear that, as expected, the convergence rate should depend on the approximability of both components of the solution. It follows that the discrete eigenvalues do not separate in two families as in the previous tests. Indeed, the two components of the solution cannot be split according to the regularity of the eigenfunctions: singular eigenfunctions correspond to a second component which is more regular and regular eigenfunctions correspond to a second component which is singular. We plan to investigate this issue further and, possibly, to extend the theory of Knyazev and Osborn [5] to mixed approximation; it would be interesting to find an example of mixed problem for which the smoothness of the eigenfunctions can be clearly individuated (see, for instance [3, Section 5.3]).

## 2. Approximation theory of multiple eigenvalues

Let $V$ and $H$ be real Hilbert spaces with $V \subset H$ with dense and continuous embedding. Let $a : V \times V \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$ be symmetric and continuous bilinear forms. We assume that $a$ is $V$-elliptic and that $b$ is equivalent to a scalar product on $H$. As usual, the hypothesis on $a$ can be weakened by assuming that a Gårding like inequality holds true. We consider the following variationally posed eigenproblem: find $\lambda \in \mathbb{R}$ such that there exists $u \in V$ with $u \neq 0$ so that

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V. \tag{1}$$

Given a sequence of finite dimensional subspaces $V_h \subset V$, with $\dim V_h = N_h$, the Galerkin approximation of problem (1) reads: find $\lambda_h \in \mathbb{R}$ such that there exists $u_h \in V_h$ with $u_h \neq 0$ so that

$$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h. \tag{2}$$

In order to analyze the approximation of discrete eigensolutions to the continuous ones, it is usual to introduce the solution operators $T : V \to V$ and $T_h : V \to V_h \subset V$ defined as follows: for $f \in V$, $Tf \in V$ is the solution of

$$a(Tf, v) = b(f, v) \quad \forall v \in V \tag{3}$$

and $T_h f \in V_h$ satisfies

$$a(T_h f, v) = b(f, v) \quad \forall v \in V_h. \tag{4}$$

It is well known that the eigenvalues $\mu$ (resp. $\mu_h$) of the operator $T$ (resp. $T_h$) are given by $\mu = 1/\lambda$ (resp. $\mu_h = 1/\lambda_h$).

We assume that $T : V \to V$ is a compact operator and that $T_h$ converges uniformly to $T$, that is

$$\|T - T_h\|_{\mathcal{L}(V)} \to 0 \quad \text{as } h \to 0. \tag{5}$$

From the compactness assumption one obtains that the eigenvalues of problem (1) form an increasing countable sequence $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, where we have repeated each eigenvalue according to its algebraic multiplicity. On the other hand problem (2) has exactly $N_h$ eigenvalues $0 < \lambda_{1,h} \leqslant \lambda_{2,h} \leqslant \cdots \leqslant \lambda_{N,h}$, each again repeated according to its algebraic multiplicity. We denote by $u_i$ the eigenfunction associated to the eigenvalue $\lambda_i$ and by $u_{i,h}$ the discrete eigenfunction associated to $\lambda_{i,h}$, and we choose $u_i$ and $u_{i,h}$ so that $b(u_i, u_i) = b(u_{i,h}, u_{i,h}) = 1$, $a(u_i, u_i) = \lambda_i$, $a(u_{i,h}, u_{i,h}) = \lambda_{i,h}$, and $b(u_i, u_j) = b(u_{i,h}, u_{j,h}) = 0$ for $i \neq j$.

The convergence in norm stated in (5) implies the convergence of eigenvalues and eigenfunctions in the following sense:

a. for any compact set $K$ contained in the resolvent set of $T$, there exists $h_0 > 0$ such that, for all $h < h_0$, $K$ is also contained in the resolvent set of $T_h$. In this case we say that there are no spurious modes;
b. if $\lambda$ is an eigenvalue of $T$ with multiplicity $m$, then there are exactly $m$ eigenvalues $\tilde{\lambda}_{1,h}, \tilde{\lambda}_{2,h}, \ldots, \tilde{\lambda}_{m,h}$ of $T_h$ (not necessarily distinct) such that $\tilde{\lambda}_{i,h}$ converges to $\lambda$ as $h$ tends to 0;
c. the gap between the direct sum of the generalized eigenspaces associated to $\tilde{\lambda}_{1,h}, \tilde{\lambda}_{2,h}, \ldots, \tilde{\lambda}_{m,h}$ and the generalized eigenspace associated to $\lambda$ tends to zero as $h$ tends to 0.

Concerning the a priori error estimates for both eigenvalues and eigenfunctions we report the classical results of the Babuška–Osborn theory (see [1] and [2] for a recent review on this theory). Let us introduce some notation. Let $\lambda \in \mathbb{R}$ be an eigenvalue of (1) with algebraic multiplicity $m$, then we denote by $E \subset V$ the eigenspace associated to $\lambda$ and by $\phi_1, \ldots, \phi_m$ a basis of eigenfunctions in $E$. Let $\tilde{\lambda}_{1,h}, \tilde{\lambda}_{2,h}, \ldots, \tilde{\lambda}_{m,h}$ be the $m$ eigenvalues of problem (2) converging to $\lambda$, we denote by $E_h \subset V_h$ the direct sum of the eigenspaces associated to $\tilde{\lambda}_{1,h}, \tilde{\lambda}_{2,h}, \ldots, \tilde{\lambda}_{m,h}$. Then the following estimates hold true:

$$\hat{\delta}(E, E_h) \leqslant C \left\| (T - T_h)_{|E} \right\|_{\mathcal{L}(V)},$$

$$\left| \frac{1}{\lambda} - \frac{1}{\tilde{\lambda}_{i,h}} \right| \leqslant \frac{1}{m} \sum_{i=1}^{m} \left| \left( (T - T_h)\phi_i, \phi_i \right)_V \right| + C \left\| (T - T_h)_{|E} \right\|^2_{\mathcal{L}(V)} \quad \text{for } i = 1, \ldots, m, \tag{6}$$