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# Two-dimensional Maxwell's equations with sign-changing coefficients

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ABSTRACT

We consider the theoretical study of time harmonic Maxwell's equations in presence of sign-changing coefficients, in a two-dimensional configuration. Classically, the problems for both the Transverse Magnetic and the Transverse Electric polarizations reduce to an equivalent scalar Helmholtz type equation. For this scalar equation, we have already studied consequences of the presence of sign-changing coefficients in previous papers, and we summarize here the main results. Then we focus on the alternative approach which relies on the two-dimensional vectorial formulations of the TM or TE problems, and we exhibit some unexpected effects of the sign-change of the coefficients. In the process, we provide new results on the scalar equations.



## 1. Introduction

The recent and promising developments of photonic metamaterials [15,16] and of plasmonics [14,1] raised new issues in the theoretical and numerical study of time harmonic Maxwell's equations: we are concerned here with the possible sign-change of the dielectric permittivity and/or of the magnetic permeability. This sign-change occurs for instance at the interface between a metal and a classical medium (vacuum or dielectric) at optical frequencies, when the metal presents a dielectric permittivity with a negligible imaginary part and a negative real part. This property is essential for the existence of plasmonic surface waves. Sign-change of the coefficients also takes place at the interface between a dielectric and a so-called left-handed metamaterial, for which both the dielectric permittivity and the magnetic permeability take negative real values (with again a small imaginary part that we neglect from now on).

We have already obtained several results related to this topic [2–5,9] for both the theoretical and the numerical aspects. In particular, we have carried out a rather thorough analysis of the corresponding scalar problem in [5]. More precisely, we have proved that the equation

$$-\operatorname{div}\left(\mu^{-1}\nabla\varphi\right) - \omega^2\varepsilon\varphi = f \tag{1}$$

in a bounded domain $\Omega$, with $f \in \mathrm{L}^2(\Omega)$ and with Dirichlet boundary conditions, may be strongly ill-posed in the usual $\mathrm{H}^1$ framework for a sign-changing function $\mu$, and we have derived conditions on $\mu$ which guarantee that the problem is of Fredholm type. The main ingredient in [5] is the so-called $\mathtt{T}$-coercivity concept, which consists in finding an isomorphism $\mathtt{T}$ of $\mathrm{H}^1_0(\Omega)$ such that the bilinear form

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$$(\varphi, \psi) \mapsto \int_{\Omega} \mu^{-1} \nabla\varphi \cdot \nabla(\mathtt{T}\psi)$$

is coercive on $\mathrm{H}^1_0(\Omega)$. The method is powerful although quite simple, since the operators $\mathtt{T}$ are built by elementary geometrical arguments.

These results have direct counterparts for Maxwell's equations in 2D configurations. In this case, it is well known that these equations give rise to two systems of equations, without any coupling, corresponding respectively to the so-called Transverse Electric (TE) and Transverse Magnetic (TM) polarizations. Moreover, each of them reduces to a scalar equation similar to (1), where $\varphi$ is the component of the electric or magnetic field parallel to the direction of invariance of the medium and of the data. Even so, the study of two-dimensional Maxwell's equations with sign-changing coefficients is interesting in its own right. Indeed, it is a preliminary step to solving two-and-a-half-dimensional electromagnetic problems which are not reducible to scalar problems, such as for instance plasmonic waveguide problems.

There is an additional motivation for choosing a vectorial formulation instead of a scalar one. As a matter of fact, numerical resolution of the TM problem using the scalar formulation provides a good approximation of the electric field, which is the unknown of the scalar problem, whereas it provides a poor approximation of the magnetic field, as it corresponds to the derivatives of the scalar unknown. On the contrary, discretizing the vectorial formulation gives an accurate approximation of the magnetic field in the appropriate norms to measure it.

Finally, from the mathematical point of view, one has to address new issues, such as compact embedding of some sets of vectorial fields. Also the proofs of the new results collected in Sections 4 and 5 are original: they use again the $\mathtt{T}$-coercivity concept but the operators $\mathtt{T}$ are no longer built from geometrical transformations as in [5]. Instead, we define them in a more abstract way, using the well-posedness of other problems. This way, we exhibit strong connections between scalar problems (1) with Dirichlet and Neumann conditions. As a consequence, a complete description of the results for the Neumann problem can be directly deduced from [5], where only the Dirichlet problem has been considered. Let us mention that parts of these results can be extended to three-dimensional Maxwell problems (see [6]).

The outline of the paper is as follows. In the next section, we briefly derive the TE and TM systems of equations and their equivalent scalar and vectorial formulations. In the sequel of the paper, we focus on the TE problem. The approach based on the scalar equation is discussed in Section 3, where we recall the main results concerning scalar transmission problems with sign-changing coefficients. Section 4 is devoted to the vectorial formulation: we introduce two hypotheses, respectively related to the Dirichlet and Neumann static scalar problems. Moreover, these two hypotheses are proved to be equivalent one to the other and, in addition, they imply a Fredholm property for the vectorial formulation. At this stage, a question remains: what happens for the vectorial approach when this hypothesis is not satisfied? This point is addressed in Section 5, where we define an appropriate variational setting.

## 2. Mathematical formulations for the transverse electric and magnetic problems

### *2.1. The equations for the TE and TM polarizations*

We consider a domain which is invariant in one direction and bounded in the transverse ones. More precisely, we introduce a domain $\Omega$ of $\mathbb{R}^2$, that is an open bounded connected set, with a connected Lipschitz boundary; then, we define $D := \{(x, y, z) \in \Omega \times \mathbb{R}\}$ and we suppose that the dielectric permittivity $\varepsilon$ and the magnetic permeability $\mu$ in $D$ are real-valued functions of $(x, y) \in \Omega$, which means that they are independent from $z$. Since we are interested by sign-changing coefficients, we suppose only that $\varepsilon \in \mathrm{L}^\infty(\Omega)$, $\mu \in \mathrm{L}^\infty(\Omega)$, $\varepsilon^{-1} \in \mathrm{L}^\infty(\Omega)$ and $\mu^{-1} \in \mathrm{L}^\infty(\Omega)$. Notice in particular that vanishing $\varepsilon$ or $\mu$ are forbidden.

In presence of a current density $\boldsymbol{J}$, the time-harmonic electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ is solution of Maxwell's equations:

$$i\omega\varepsilon \boldsymbol{E} + \mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \quad \text{and} \quad -i\omega\mu \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = 0 \quad \text{in } D, \tag{2}$$

where a time behavior in $e^{-i\omega t}$ is assumed, $\omega > 0$. We suppose moreover that $D$ is bounded by a perfect conductor, so that the tangential trace of $\boldsymbol{E}$, and therefore the normal trace of $\mu\boldsymbol{H}$, both vanish on $\partial D$:

$$\boldsymbol{E} \times \boldsymbol{\nu} = 0 \quad \text{and} \quad \mu\boldsymbol{H} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial D \tag{3}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal vector field to $\partial D$.

**Remark 2.1.** Using (2) and (3), one finds easily that $\varepsilon^{-1}(\mathbf{curl}\, \boldsymbol{H} - \boldsymbol{J}) \times \boldsymbol{\nu} = 0$ on $\partial D$.

Assuming finally that the current density $\boldsymbol{J}$ is independent from $z$, the problem becomes completely independent from $z$ and the simplification $\frac{\partial \cdot}{\partial z} = 0$ leads to the following expanded equations where we have set $\boldsymbol{J} := (J_x, J_y, J_z)^t$, $\boldsymbol{E} := (E_x, E_y, E_z)^t$ and $\boldsymbol{H} := (H_x, H_y, H_z)^t$: