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On quasi-static models hidden in Maxwell's equations [☆]

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ABSTRACT

In this paper we introduce the electromagnetic quasi-static models in a simple but meaningful way, relying on the dimensional analysis of Maxwell's equations. This analysis puts in evidence the three characteristic times of an electromagnetic phenomenon. It allows to define the range of validity of well-known models, such as the eddy-current (MQS) or the electroquasistatic (EQS) ones, and thus their pertinence to describe a given phenomenon. The role of the so-called "small parameters" of a model is explained in detail for two classical examples, namely a capacitor and a solenoid. We show how the MQS and EQS models result from having replaced fields by their first order truncations of Taylor expansions with respect to these small parameters. We finally investigate the connection between quasi-static models and circuit theory, clarifying the role of the fields with respect to classical circuit elements, and provide an example of application to study the electromagnetic fields in a simple case.

1. Introduction

Maxwell's equations (see for example [10]) are fundamental for the description of electromagnetic phenomena and valid over a wide range of spatial and temporal scales. The static limit of the theory is well defined. The electric and magnetic fields are given by the laws of Coulomb and Biot–Savart. As soon as there is any time dependence, we should in principle use the full set of Maxwell's equations with all their complexity. However, concrete problems in electromagnetism rarely require the solution of Maxwell's equations in full generality, because of various simplifications due to the smallness of some terms [4]. In this work, we try to quantify this smallness by means of a dimensional analysis [2,5] of Maxwell's equations. Indeed, some particular models in the low frequency limit, also known as quasi-static range (QS), emerge from neglecting particular couplings of electric and magnetic field related quantities. Following [12–14], we discuss the fact that there exists not one but indeed two dual Galilean limits called "electric" or EQS, and "magnetic" or MQS limits, the first including capacitive effects, the latter inductive effects. A dimensional analysis on the fields allows to emphasize the correct scaling yielding the two (limit) sets of Maxwell's equations. By means of detailed mathematical steps for two classical physical situations, we underline the role in the description of the fields' amplitude of the "small parameters" resulting from the dimensional analysis of Maxwell's equations. We provide simple numerical results on equivalent electric circuits to support the conclusions on the time-range validity of the considered quasi-static models. In some concrete applications however, at a certain frequency and for certain configurations of inductors, the separation between inductive and capacitive effects is not possible (see an example in [6]). In these cases, suitable formulations have then to be designed on the basis of the ones discussed in this work. The present work aims at proposing a new derivation of some existing quasi-static

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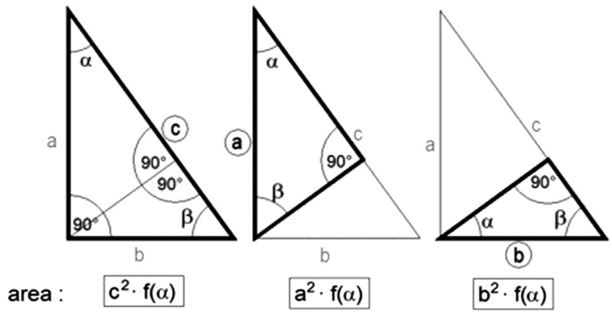

Fig. 1. The area of any triangle T depends on its size and shape which are defined by giving one edge, the largest c, and two angles α, β (for straight triangles, one angle, say α, is enough as $\beta = 90^\circ - \alpha$). Thus straight triangles with thick perimeter have area $c^2 f(\alpha)$ (left), $a^2 f(\alpha)$ (center), $b^2 f(\alpha)$ (right), respectively, with f a non-dimensional function of the angles which are dimensionless quantities too. Divide T into 2 non-overlapping smaller triangles T_1, T_2, by using the perpendicular to c as indicated, then $\text{area}(T) = \text{area}(T_1) + \text{area}(T_2)$ that yields $c^2 f(\alpha) = a^2 f(\alpha) + b^2 f(\alpha)$. Eliminating f, one gets $c^2 = a^2 + b^2$ without never specifying the form of f.

models by means of a dimensional analysis of Maxwell's equations. Moreover, we provide a condition stated on the basis of (physical) quantities related to the phenomenon, which allows to identify the mathematical model best suited for its investigation. To be more precise, the emphasis is put on the following points which cover both mathematical modeling and numerical validation. In Section 2 we recall the basis of dimensional analysis and we apply it in Section 3 to introduce the characteristic times for an electromagnetic phenomenon. In Section 4 we recall that the quasi-static models are Galilean limits of Maxwell's equations and underline that the field amplitude ratio matters in the selection of a limit. In Section 5, we scale Maxwell's equations using non-dimensional quantities naturally related to the previously introduced characteristic times and amplitude ratios, and state the model to be used according to the scaling. The mathematical part ends in Section 6 with a justification of the parameters introduced so far as the natural ones that appear when performing an asymptotic analysis of Maxwell's equations for two classical applications. Finally, in Section 7 we analyze the connection between quasi-static models and RCL circuits, and a numerical validation of the presented models is proposed in Section 8 together with some concluding remarks.

2. Dimensional analysis: known concepts

In physics and other sciences, we have to deal with distances or time intervals that, in a Galilean perspective, we are able to measure, comparing the first with a graduated meter and the second on a suitable clock. When we measure a quantity $\mathbf{g}$ with respect to a unit $\mathbf{u}$ we write it as $\mathbf{g} = g\mathbf{u}$ with g a real number. The surface of a (planar) square with side of size ℓ with respect to a fixed unit $\mathbf{u}$ is $\mathbf{a} = \ell^2\mathbf{u}^2$. The numbers g, ℓ, are however approximate due to errors in the measurement process, better if we express the measure of $\mathbf{g}$ or of $\mathbf{a}$ with respect to another quantity of the same kind chosen as unit. In this way, we introduce the dimension L of lengths and say that the surface has the dimension of the square of a length by writing

$$[\mathbf{a}] = [L]^2. \tag{1}$$

The unit of a physical quantity and its dimension are linked, but not identical concepts. The units of a physical quantity are defined by convention and related to some standard; *e.g.*, length may have units of meters, feet, inches, miles or micrometers; but any length always has the dimension of L, independent of what units are arbitrarily chosen to measure it. The concept of dimension is thus more abstract than that of unit: length is a dimension and meter is a unit $\mathbf{u}$ for lengths. If we change the system of units $\mathbf{u}$ for the length setting $\mathbf{u}' = \lambda\mathbf{u}$ then the surface in the new system of units becomes $\mathbf{a}' = \lambda^{-2}\mathbf{a}$ but still $[\mathbf{a}'] = [L]^2$. Similarly, for a volume we have $[L]^3$. In the case of angles θ, since their measure is expressed as ratio between lengths, we have $[\theta] = [L]^0$, thus angles have no dimension and the same for all trigonometric functions of angles. See Fig. 1 for a proof of the Pythagorean theorem in the Euclidean plane on the basis of these few concepts.

The main difficulty in dimensional analysis is the selection of the physical dimensions. First Isaac Newton (1686), who referred to it as the Great Principle of Similitude, then James Clerk Maxwell (1855) played a major role in establishing modern use of dimensional analysis by distinguishing mass M, length L, time T and current intensity I as fundamental quantities, while referring to others as derived (other quantities are also considered as fundamental but will not be involved in what follows). So, for example, by writing $[v] = [M]^0[L]^1[T]^{-1}[I]^0$ we say that speed has the dimension of a length divided by a time in any possible system of units. The selection of the fundamental dimensions – why the current I instead of the tension V which is easier to measure? – is a largely discussed subject in the literature and goes beyond the purpose of the present work.

Another important step is the idea that physical laws, such as force equals mass times acceleration, must be independent of the units used to measure the involved physical variables, here mass and acceleration. This led to the conclusion that meaningful laws must be homogeneous equations in their various systems of units, a result which was later formalized in the Vashy–Buckingham theorem [5]. Indeed, this theorem describes how every physically meaningful equation involving