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Estimator reduction and convergence of adaptive BEM

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Dedicated to Professor George Hsiao on the occasion of his 75th birthday

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A B S T R A C T

A posteriori error estimation and related adaptive mesh-refining algorithms have themselves proven to be powerful tools in nowadays scientific computing. Contrary to adaptive finite element methods, convergence of adaptive boundary element schemes is, however, widely open. We propose a relaxed notion of convergence of adaptive boundary element schemes. Instead of asking for convergence of the error to zero, we only aim to prove estimator convergence in the sense that the adaptive algorithm drives the underlying error estimator to zero. We observe that certain error estimators satisfy an estimator reduction property which is sufficient for estimator convergence. The elementary analysis is only based on Dörfler marking and inverse estimates, but not on reliability and efficiency of the error estimator at hand. In particular, our approach gives a first mathematical justification for the proposed steering of anisotropic mesh-refinements, which is mandatory for optimal convergence behavior in 3D boundary element computations.

1. Introduction

1.1. Convergence of adaptive algorithms

In many applications, numerical simulations are based on a triangulation $\mathcal{T}_\ell := \{T_1, \dots, T_N\}$ of the simulation domain. Let the (unknown) exact solution u belong to a certain Hilbert space $\mathcal{H}$ with norm $|||\cdot|||$. Then, for some discrete subspace X_ℓ of $\mathcal{H}$ associated with $\mathcal{T}_\ell$, a numerical approximation $u_\ell \in X_\ell$ is computed. Refinement of $\mathcal{T}_\ell$ yields an improved approximation. Usually, in the context of boundary integral equations, u has certain singularities so that uniform mesh-refinement leads to a poor convergence behavior for the error $|||u - u_\ell|||$. Contrary, adaptive algorithms have themselves proven to provide an effective means to improve the accuracy of u_ℓ. Based on the local contributions $\rho_\ell(T_j)$ of an a posteriori error estimator ρ_ℓ, these algorithms only refine certain elements $T_j \in \mathcal{T}_\ell$, where the error appears to be large. Starting from an initial mesh $\mathcal{T}_0$, this procedure generates a sequence of triangulations $\mathcal{T}_\ell$ and corresponding discrete solutions $u_\ell \in X_\ell$. However, since adaptive mesh-refinement does not guarantee that

$$\max_{T_j \in \mathcal{T}_\ell} \operatorname{diam}(T_j) \xrightarrow{\ell\to\infty} 0, \tag{1}$$

in general, the verification of *convergence*

$$|||u - u_\ell||| \xrightarrow{\ell\to\infty} 0 \tag{2}$$

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is a nontrivial issue. Whereas (2) is well-studied for adaptive finite element methods (AFEM), see [15,25,31] and the references therein, this question is essentially open for adaptive boundary element methods (ABEM), where only preliminary convergence results [14,21] are available.

1.2. Concept of estimator reduction

We aim at contributing to the mathematical understanding of h-adaptive BEM. To that end, we primarily ask for *estimator convergence*

$$\rho_\ell \xrightarrow{\ell\to\infty} 0. \tag{3}$$

For certain estimators from the BEM literature, we prove that the marking criterion from [16] guarantees some *estimator reduction*

$$\rho_{\ell+1} \leqslant q\rho_\ell + C|||u_{\ell+1} - u_\ell||| \quad \text{for all } \ell \in \mathbb{N} \tag{4}$$

with ℓ-independent constants $0 < q < 1$ and $C > 0$. If the sequence u_ℓ of discrete solutions is convergent to some (unknown) limit

$$u_\infty := \lim_{\ell\to\infty} u_\ell \in \mathcal{H}, \tag{5}$$

as is the case in usual adaptive Galerkin schemes, the estimator reduction (4) already implies the estimator convergence (3). If furthermore the estimator ρ_ℓ provides some upper bound for the error $|||u - u_\ell|||$, this yields convergence (2) and, in particular, $u = u_\infty$.

1.3. Main results & outline

In Section 2, we state our version of the adaptive algorithm (Algorithm 2.1), observe that adaptive Galerkin BEM always guarantees the a priori convergence (5), and prove that the estimator reduction (4) thus implies the estimator convergence (3). In the remainder of this work, the weakly-singular integral equation for the 2D and 3D Laplacian serves as model problem. This and the lowest-order Galerkin BEM are stated in Section 3. We then focus on $(h - h/2)$-type estimators from [20] and averaging estimators from [11,13]. In Section 4, we observe that isotropic mesh-refinement steered by these estimators guarantees (4), see Theorems 4.1 and 4.3. In 3D BEM, however, anisotropic mesh-refinement is, in general, necessary to resolve edge singularities effectively. Even in the context of FEM, there are — to the best of our knowledge — no rigorous convergence results for adaptive Galerkin schemes with anisotropic mesh-refinement. In Section 5, we consider a heuristics from [20] to steer anisotropic adaptive mesh-refinement. First, this idea is generalized from the $(h - h/2)$-error estimator to the averaging error estimators. Second, we prove that the proposed adaptive schemes again guarantee the estimator reduction (4), see Theorems 5.1 and 5.2. This means that the concept of estimator reduction gives a mathematical justification for the anisotropic refinement criterion used and allows for a first convergence result of adaptive anisotropic 3D BEM. Numerical experiments included in Section 4 and Section 5 underline our theoretical findings.

1.4. Some remarks

We stress that the verification of estimator reduction (4) in Theorems 4.1, 4.3, 5.1, and 5.2 depends only on the definition of the local mesh-refinement and on a local inverse estimate. Moreover, our analysis applies to a quite general class of local mesh-refinement rules, e.g., any rule based on newest-vertex bisection or even anisotropic mesh-refinement with rectangular elements. In particular, the proof of estimator convergence (3) does neither use reliability, nor efficiency of the error estimator ρ_ℓ at hand, i.e., estimator convergence is independent of whether ρ_ℓ provides a lower or upper bound for $|||u - u_\ell|||$.

It came as a surprise to us that a convergence result (2) for adaptive Galerkin schemes can thus be obtained by an elementary and straight-forward analysis. In particular, we see that convergence (2) relies only on the reliability of ρ_ℓ, but not on (discrete local) efficiency as used e.g. in [24,25] in the context of AFEM.

1.5. Possible generalizations & extensions

The results of this paper, although stated for the weakly-singular integral equation, also apply to other elliptic integral equations and the corresponding $(h - h/2)$ or averaging error estimators; we refer to [19] for $(h - h/2)$-based error estimators and to [12,13] for averaging error estimators for some hypersingular integral equation in 2D. Moreover, the concept of estimator reduction provides a general framework for convergence analysis. Further applications read as follows:

First, in BEM computations, it is usually necessary to discretize the given data to work with discrete integral operators only. If the data are discretized by projecting them appropriately, this provides some a priori convergence of the data. Although the discrete solutions u_ℓ are then computed with respect to different right-hand sides, this allows to prove a priori