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Convergence of adaptive BEM for some mixed boundary value problem

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ABSTRACT

For a boundary integral formulation of the 2D Laplace equation with mixed boundary conditions, we consider an adaptive Galerkin BEM based on an $(h-h/2)$-type error estimator. We include the resolution of the Dirichlet, Neumann, and volume data into the adaptive algorithm. In particular, an implementation of the developed algorithm has only to deal with discrete integral operators. We prove that the proposed adaptive scheme leads to a sequence of discrete solutions, for which the corresponding error estimators tend to zero. Under a saturation assumption for the non-perturbed problem which is observed empirically, the sequence of discrete solutions thus converges to the exact solution in the energy norm.

1. Introduction

The $(h-h/2)$-error estimation strategy is a well-known technique to derive a posteriori estimators for the error $|||\mathbf{u}-\mathbf{U}_\ell|||$ in the energy norm; see [22] in the context of ordinary differential equations, and the overview article of Bank [5] or the monograph [1, Chapter 5] in the context of the finite element method: Let $\mathcal{X}_\ell$ be a discrete subspace of the energy space $\mathcal{H}$ and let $\widehat{\mathcal{X}}_\ell$ be its uniform refinement. With the corresponding Galerkin solution $\mathbf{U}_\ell$ and $\widehat{\mathbf{U}}_\ell$, the canonical $(h-h/2)$-*error estimator*

$$\eta_\ell := |||\widehat{\mathbf{U}}_\ell - \mathbf{U}_\ell||| \tag{1.1}$$

is a computable quantity [14] which can be used to estimate $|||\mathbf{u}-\mathbf{U}_\ell|||$, where $\mathbf{u}\in\mathcal{H}$ denotes the exact solution.

For finite element methods (FEM), the energy norm, e.g., $|||\cdot||| = \|\nabla(\cdot)\|_{L^2(\Omega)}$ provides local information, which elements of the underlying mesh should be refined to decrease the error effectively. For boundary element methods (BEM), the energy norm $|||\cdot|||$ is (equivalent to) a fractional-order (and possibly negative) Sobolev norm and typically does not provide local information. In [17,19], localized variants of η_ℓ were introduced for certain weakly-singular and hypersingular integral equations. In [16,17] the equivalence of η_ℓ to hierarchical two-level error estimators from [23,26,29] and averaging error estimators from [9–11] has been proven.

Recently [18], convergence of some $(h-h/2)$-steered adaptive mesh-refinement has been proven for linear model problems in the context of FEM and BEM. In [3], the concept of estimator reduction has been introduced to analyze convergence of anisotropic mesh-refinement steered by $(h-h/2)$-type or averaging-based error estimators for weakly-singular integral equations arising in 3D BEM. However, in [3,18] it is assumed that the right-hand side of the integral equation is computed

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analytically. This assumption is relaxed in [4] where, for the Dirichlet problem, the resolution of the given data becomes a part of the adaptive loop.

In this work, we consider the so-called symmetric integral formulation of a mixed boundary value problem in 2D. Contrary to prior works, we include the approximation of the given Dirichlet, Neumann, and volume data into the a posteriori error estimate. Therefore, the proposed scheme deals with discrete integral operators only which can then be approximated by means of hierarchical matrices [21] or the fast multipole method, cf. [30] and the references therein.

The proposed error estimator ϱ_ℓ is a sum of certain $(h - h/2)$-error estimators which control the discretization error, and certain data oscillation terms which control the consistency errors introduced by the data approximation. The estimator ϱ_ℓ is easily implemented and can be computed in linear complexity. In particular, it is part of the developed MATLAB BEM library HILBERT [2].

Using the concept of estimator reduction from [3], we even prove that the usual adaptive algorithm, steered by ϱ_ℓ, enforces $\lim_\ell \varrho_\ell = 0$. Under a saturation assumption, which is empirically observed in numerical experiments, this implies convergence $\lim_\ell \mathbf{U}_\ell = \mathbf{u}$ of the discrete solutions. In [12], a safeguard strategy checks whether the saturation assumption fails. In this case, uniform refinement is used in the corresponding step of the adaptive loop and convergence of $\mathbf{U}_\ell$ to $\mathbf{u}$ is mathematically guaranteed. The analysis of [12], however, is only complete if the right-hand side of the integral equation is computed analytically.

The remainder of this paper is organized as follows: Section 2 introduces the so-called symmetric integral formulation of the model problem as well as the data-perturbed Galerkin formulation and states the main results of this work, where we first focus on homogeneous volume forces. In Section 3, we collect the essential preliminaries, namely the mapping properties of the involved integral operators as well as certain inverse estimates and approximation results. Section 4 introduces and analyzes the data oscillation terms as well as the proposed error estimator ϱ_ℓ. Our version of the adaptive mesh-refining algorithm is found in Section 5, and we prove convergence $\lim_\ell \varrho_\ell = 0$. In Section 6, we briefly sketch how the analysis can be generalized to non-homogeneous volume forces. Numerical experiments in Section 7 underline that the proposed adaptive algorithm performs very effectively in practise.

2. Model problem and analytical results

The aim of this section is to introduce the model problem, its integral formulation, and the Galerkin formulation. Moreover, we sketch our main results and give and overview on the results contained in this work.

2.1. Continuous model problem

In a first step, we consider the homogeneous mixed boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \partial_n u = \phi_N & \text{on } \Gamma_N, \end{cases} \tag{2.1}$$

where $u_D : \Gamma_D \to \mathbb{R}$ and $\phi_N : \Gamma_N \to \mathbb{R}$ are given data and where the solution $u : \Omega \to \mathbb{R}$ is sought. We assume that the boundary $\Gamma := \partial\Omega$ of the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ is polygonal and split into (relatively open) Dirichlet and Neumann boundaries, which satisfy $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, and $|\Gamma_D| > 0$. For technical reasons, we further assume $\operatorname{diam}(\Omega) < 1$, see Section 3.2 below. This problem is equivalently recast into the so-called symmetric boundary integral formulation

$$A \begin{pmatrix} u_N \\ \phi_D \end{pmatrix} = (1/2 - A) \begin{pmatrix} u_D \\ \phi_N \end{pmatrix} =: F \quad \text{with operator matrix } A := \begin{pmatrix} -K & V \\ W & K' \end{pmatrix}, \tag{2.2}$$

see e.g. [32, Section 3.4.2] or [34, Section 7.3]. Here, V is the simple-layer potential, K is the double-layer potential with adjoint K', and W is the hypersingular integral operator, which are defined in Sections 3.2–3.4 below. In this formulation, we fix arbitrary extensions $u_D : \Gamma \to \mathbb{R}$ and $\phi_N : \Gamma \to \mathbb{R}$ which satisfy $(u_D, \phi_N) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and seek a solution $\mathbf{u} := (u_N, \phi_D) \in \mathcal{H} := \widetilde{H}^{1/2}(\Gamma_N) \times \widetilde{H}^{-1/2}(\Gamma_D)$.

Let $\langle \cdot, \cdot \rangle$ denote the $L^2(\Gamma)$-scalar product which is extended to duality between $\widetilde{H}^{-1/2}(\Gamma_D)$ and $H^{1/2}(\Gamma_D)$ and between $H^{-1/2}(\Gamma_N)$ and $\widetilde{H}^{1/2}(\Gamma_N)$. Note that A is a linear and continuous operator from $\mathcal{H}$ to $\mathcal{H}^* = H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_N)$, where duality is understood via

$$\big\langle (v_D, \psi_N), (v_N, \psi_D) \big\rangle_{\mathcal{H}^* \times \mathcal{H}} := \langle \psi_N, v_N \rangle + \langle \psi_D, v_D \rangle \quad \text{for all } (v_N, \psi_D) \in \mathcal{H}, (v_D, \psi_N) \in \mathcal{H}^*. \tag{2.3}$$

In particular, A induces a continuous bilinear form on $\mathcal{H}$, namely

$$\begin{aligned} \big\langle\!\big\langle (u_N, \phi_D), (v_N, \psi_D) \big\rangle\!\big\rangle &:= \big\langle A(u_N, \phi_D), (v_N, \psi_D) \big\rangle_{\mathcal{H}^* \times \mathcal{H}} \\ &= \big\langle W u_N + K' \phi_D, v_N \big\rangle + \langle \psi_D, -K u_N + V \phi_D \rangle. \end{aligned} \tag{2.4}$$