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# Stokes, Maxwell and Darcy: A single finite element approximation for three model problems

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ABSTRACT

In this work we propose stabilized finite element methods for Stokes', Maxwell's and Darcy's problems that accommodate any interpolation of velocities and pressures. We briefly review the formulations we have proposed for these three problems independently in a unified manner, stressing the advantages of our approach. In particular, for Darcy's problem we are able to design stabilized methods that yield optimal convergence both for the primal and the dual problems. In the case of Maxwell's problem, the formulation we propose allows one to use continuous finite element interpolations that converge optimally to the continuous solution even if it is non-smooth. Once the formulation is presented for the three model problems independently, we also show how it can be used for a problem that combines all the operators of the independent problems. Stability and convergence is achieved regardless of the fact that any of these operators dominates the others, a feature not possible for the methods of which we are aware.



## 1. Introduction

The numerical approximation of partial differential equations (PDEs) in general geometries can be performed by using finite element (FE) techniques. The standard approach to the problem consists of considering the weak form of the PDE, and replace the infinite dimensional functional spaces for the solution and test functions by finite dimensional ones. Those finite dimensional spaces are constructed using FE functions over a partition of the domain.

PDEs defined by coercive differential operators can be approximated by the Galerkin FE technique, provided that the corresponding FE space can approximate functions in the continuous functional space; coercivity of the continuous problem is inherited by the discrete one. However, PDEs that exhibit a saddle-point structure, and so stability is attained via a (less demanding) inf–sup condition, cannot be straightforwardly approximated by only looking at the approximability properties of the FE space. The reason is quite simple: inf–sup conditions satisfied by the continuous problem are not inherited (in general) by their discrete versions. Therefore, FE spaces are not only required to exhibit an approximability property, but also a discrete inf–sup condition.

Saddle-point problems include the primal unknown and the dual one, the Lagrange multiplier. FE pairs for these unknowns have to be built such that they satisfy a discrete inf–sup condition (see, e.g., [8]). Examples of linear PDEs with this structure are Stokes' problem, Darcy's problem and Maxwell's problem. Every problem involves a different differential operator, and their well-posedness relies on different inf–sup conditions. It is not surprising that stable FE approximations (called inf–sup stable) are different from one problem to the other. Using inf–sup stable FE methods, e.g. the Stokes problem could be approximated by the Crouzeix–Raviart element [15], Darcy's problem (in dual form) would be solved by using the

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Raviart–Thomas FE [23], whereas Maxwell's problem would be discretized by using Nédélec elements [21,22]. More recently, inf–sup stable FEs for these problems have been nicely casted in the frame of de Rham sequences; see [1,2] for details. Even though this approach can be appealing when we want to solve one of these problems alone, it is not suitable for multi-physics simulations that couple different operators. The FE spaces for every sub-problem are different, and the unknowns are evaluated in different ways; it complicates the implementation, mainly the data-structure and the integration of the coupling terms. Furthermore, when these operators are combined with convection terms, like the Navier–Stokes equations, Galerkin FE techniques exhibit instabilities in the singular limit of dominant convection.

Alternatively, we can consider stabilized FE methods. The idea is to introduce additional terms to those obtained from the Galerkin technique that will provide the desired stability without the need to satisfy a discrete inf–sup condition. Obviously, we want these methods not to spoil the convergence of the Galerkin technique; this is usually attained by the introduction of residual-based terms that also make the final system consistent. However, we can also consider non-consistent but optimal techniques. Stabilized FE methods were originally motivated for the stabilization of the convection-diffusion equation in the convection dominant regime [9]. Some time later, these techniques were proved to be effective also for the stabilization of the pressure in the Stokes problem, allowing to avoid the satisfaction of the inf–sup condition (see [19]). Many years later, these ideas were extended to the Darcy problem in primal form in [20]. Then, a stabilized FE technique for the dual Darcy problem that exhibits the same convergence rates as inf–sup stable FEs was proposed in [3,4]. Very recently, a stabilized FE formulation for the Maxwell problems that allows to use Lagrange finite element methods and converge also to singular solutions has been designed in [5]. Using the stabilized FE approach, all the unknowns for all these problems can be approximated via Lagrangian (nodal) FE spaces. This approach is clearly well-suited for multiphysics, since we can consider a simple data structure, the integration of all the terms involve the same FE spaces, and all the unknowns are defined in the same way. Further, it allows to use computationally efficient nodal FEs. The aim of this work is to show for the first time that the Stokes, Maxwell and Darcy problems can be treated in a unified way. As a result, we can consider numerical methods for the combined Stokes–Maxwell–Darcy problem whose stability is independent of the physical parameters, something that cannot be attained by inf–sup stable finite elements satisfying a discrete de Rham sequence, since every problem requires a different discretization.

## 2. Model problems

In this section we present the finite element approximation we propose for the Stokes, the Maxwell and the Darcy problems *separately*. After stating the problems, we discuss their functional framework, which has direct consequences on the numerical approximation. The Galerkin approximation is presented then, and the stabilized formulations we propose follow. Our objective is to show which is the stabilization mechanism in each case.

### *2.1. Boundary value problems*

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be the domain where the problem needs to be solved. The problems we are interested in consist in finding a vector field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a scalar field $p : \Omega \longrightarrow \mathbb{R}$ such that

*Stokes' problem*

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad & \text{on } \partial\Omega. \end{aligned}$$

*Maxwell's problem*

$$\begin{aligned} \lambda \nabla \times \nabla \times \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{0} \quad & \text{on } \partial\Omega. \end{aligned}$$

*Darcy's problem*

$$\begin{aligned} \sigma \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = g \quad & \text{in } \Omega, \\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

In these equations, $\boldsymbol{f}$ is the vectors of body forces, $g$ is a given mass flow, and $\nu$, $\lambda$ and $\sigma$ are physical parameters. In general, for the Stokes and Maxwell problems, $g$ is considered zero. This is the case considered above and when analyzing