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A numerical treatment of wet/dry zones in well-balanced hybrid schemes for shallow water flow

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ABSTRACT

The flux-limiting technology that leads to hybrid, high resolution shock capturing schemes for homogeneous conservation laws has been successfully adapted to the non-homogeneous case by the second and third authors. In dealing with balance laws, a key issue is that of well-balancing, which can be achieved in a rather systematic way by considering the 'homogeneous form' of the balance law.

The application of these techniques to the shallow water system requires also an appropriate numerical treatment for the wetting/drying interfaces that appear initially or as a result of the flow evolution. In this paper we propose a numerical treatment for wet/dry interfaces that is specifically designed for schemes based on the 'homogeneous form'. We also show that it maintains the well-balancing properties of the underlying hybrid schemes.

1. Introduction

The shallow water equations are widely used in the modeling of a number of real life applications ranging from ocean and hydraulic engineering to the modeling of tidal flows in estuary and coastal areas, bore wave propagation, as well as river, reservoir, and open channel flows, among others. This system of partial differential equations is derived from the Navier–Stokes equations, neglecting diffusion of momentum by viscous and turbulent effects and not including wind effects and Coriolis force terms. Ignoring also friction losses, the resulting system describes the flow as a system of conservation laws with additional source terms that model only the effect of the bottom topography, or bathymetry.

Research on numerical methods for the solution of the shallow water system has attracted much attention in recent years (see, e.g. [6,8,12,23,27,26] and references therein). The system admits stationary solutions in which nonzero flux gradients are exactly balanced by the source terms, and very often these solutions, or solutions near these steady states, need to be accurately computed. In such cases, numerical schemes are also required to satisfy a discrete equivalent of this flux-gradient/source-term balancing that occurs at steady states, otherwise numerically generated oscillations might obscure the true solution being sought. This requirement has given rise to the concept of *Well Balanced* (WB) schemes, which was independently introduced by Greenberg and Leroux in [14], and by Bermúdez and Vázquez-Cendón in [3], where they proposed the so-called C-property in a numerical scheme as the key to achieve the necessary discrete balance.

Well balanced schemes often involve a specialized treatment, which may include the use of exact or approximate Riemann solvers for the shallow water system (see e.g. [11,12]). On the other hand, a different approach to the discrete balance required by WB schemes was considered by Gascón and Corberán in [13], in the context of duct flow. This approach is based

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on exploiting a formal transformation of the non-homogeneous problem to a 'homogeneous form' through the definition of a 'combined flux', obtained by adding the convective flux to a primitive of the source term. The 'homogeneous form' allows Gascón and Corberán to devise a TVD strategy to compute numerical approximations to the Euler equations with the source terms of geometrical nature appearing in duct flow simulations.

In [17,9], we follow a different strategy in order to obtain a high resolution numerical scheme for balance laws. Considering a general scalar balance law, as in [14,15], we realize that the TVD construction in [13] must, necessarily, be of restricted use in this case, since there is no guarantee that the true solution of the scalar balance law has this property. However, the potential for *automatic well-balancing* that is inherent to the 'homogeneous form', allows us to directly use a Lax–Wendroff approach in order to construct a second-order well-balanced numerical scheme for general balance laws. Our WB2 (for well-balanced second order) scheme, has all the expected properties, including the typical Gibbs-like oscillatory behavior around discontinuities in the solution which characterize Lax–Wendroff schemes for homogeneous conservation laws. The main goal of the TVD technology used by Gascón and Corberán in [13] was, precisely, to curb this oscillatory behavior.

In [17,9], we propose to adapt instead the flux-limiting technology, well-established for homogeneous conservation laws, in order to construct a high resolution shock capturing scheme for general balance laws. As shown in [9], the hybridization process needs to be carried out in such a way that the well-balancing of the resulting scheme is respected. The numerical examples on scalar models in [9] show the importance of this issue, in particular for the scalar models of shallow water flow studied by Greenberg and Leroux in [14]. A key feature of our construction is that it identifies the key issues in the numerical technique leading to the WB property, which allows us to design a hybridization process between the WB2 scheme and a first order, well-balanced, non-oscillatory counterpart. In [17,18], we show how to carry out the extension of this technique to nonlinear systems of balance laws in one dimension. The numerical experiments shown in [17,18] demonstrate that the hybrid scheme has the expected properties.

An important difficulty arising in the simulation of free surface flows is the appearance of dry areas, either due to the initial conditions or as a result of the fluid motion. If no modifications are made, standard numerical schemes fail in the presence of wet/dry situations, producing spurious results. The implementation of a proper treatment that can handle wet/dry interfaces and dry areas in a given numerical technique is by no means a trivial task, as many difficulties appear. In particular, the numerical fluxes must be modified according to the kind of wet/dry transition found, and the resulting scheme has to maintain the WB property. Several methods can be found in the literature which overcome this problem for various numerical techniques (see [4,25]). In this work we describe a rather straightforward modification, specifically designed to handle wet/dry transitions in our numerical scheme. The treatment combines results in [6] and [7] and, according to our numerical experiments, it leads to a robust numerical scheme that can appropriately handle the occurrence of existing or forming wet/dry fronts in numerical simulations.

The paper is organized as follows: Section 2 is devoted to recalling the construction of the WB2 scheme: a second order, Lax–Wendroff type scheme based on the homogeneous form of the balance law. We also recap the results concerning the well-balancing properties of the scheme, as well as the extension to 1D systems. In Section 3, we briefly review our implementation of the WBH2 (well-balanced, hybrid, second order) scheme proposed in [9] and its extension to 1D systems of hyperbolic balance laws. In Section 4, we concentrate on the shallow water system and on the specific details of the scheme for 1D shallow water flows. In Section 5 we describe the numerical treatment implemented at wet/dry fronts and at interfaces where it is suspected that the generation of a dry area might occur. Finally, in Section 6, we perform several numerical experiments that confirm our theoretical observations and demonstrate the robustness of the scheme. We close the paper with some conclusions and perspectives for future work in Section 7.

2. The WB2 scheme in 1D

2.1. The scalar case

In [13], Gascón and Corberán propose to construct a TVD scheme for balance laws by considering the so-called *homogeneous form* of the balance law

$$\text{(a)} \quad u_t + f(u)_x = s(x,u) \quad \equiv \quad \text{(b)} \quad u_t + g_x = 0. \tag{1}$$

The equivalence between (1)(a) and (1)(b) is ensured if the function g satisfies $g_x = f_x - s$, which can be easily achieved by formally defining $g = f - \int^x s$.

Inspired by the work in [13], we derived in [9] a Lax–Wendroff type second-order, one-step, numerical scheme. Our derivation was based on considering a direct dependence of the combined flux, g, in terms of the space and time variables. We briefly recall here the derivation of this scheme for the sake of completeness. The reader is referred to [9] for full details on the scheme.

We formally seek a Lax–Wendroff-type discretization of (1)(b), hence

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\big(G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}\big), \tag{2}$$

$$\frac{1}{\Delta x}\big(G^n_{i+\frac{1}{2}} - G^n_{i-\frac{1}{2}}\big) = (g_x)\big|_i^{n+1/2} + O\big(\Delta x^2 + \Delta t^2\big). \tag{3}$$