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A least-squares fem-bem coupling method for linear elasticity

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ABSTRACT

This paper deals with a least-squares formulation of a second order transmission problem for linear elasticity. The problem in the unbounded exterior domain is rewritten with boundary integral equations on the boundary of the inner domain. In the interior domain we treat a linear elastic material which can also be nearly incompressible. The least-squares functional is given in terms of the $\tilde{\mathbf{H}}^{-1}(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$ norms. These norms are realized by solution operators of corresponding dual norm problems which are approximated using multilevel preconditioners.

1. Introduction

The coupling of finite elements and boundary elements has become a popular and powerful method for the solution of transmission problems of different types. Also the use of mixed finite elements instead of the standard FEM for treating e.g. mechanical problems has been increased in the last years. With the mixed finite element method more accurate computations of the stresses can be obtained. Therefore research on the combination of mixed finite element methods and either boundary integral equations or Dirichlet-to-Neumann (DtN) mappings increased (see e.g. [16,17] and [9]), where boundary integral equations or DtN mappings can be applied to linear homogeneous materials in bounded or unbounded domains. The main difficulty in the usage of mixed FEM/BEM coupling methods is to find appropriate finite element spaces which satisfy discrete inf-sup conditions to assure the existence of a unique discrete solution and the stability of the method.

It is known that least-squares methods need not to fulfill a discrete inf-sup condition and hence they are very interesting for mixed formulations. Our numerical experiments show that the least square coupling method, applied to model incompressible material, is locking-free when standard finite elements are used – contrary to mixed methods. The investigation of least-squares methods started with the work of Bramble and Schatz [6]. Afterwards many authors studied the use of variational formulations of least-squares type (see e.g. [3]). One main least-squares approach was introduced by Stephan and Wendland [23], Wendland [25], Jesperson [21] and Aziz et al. [1] where the general theory of elliptic boundary value problems of Agmon–Douglas–Nirenberg type was used and leads to a system of minimization of a least-squares functional consisting of weighted residuals of differential equations and boundary conditions. Another approach was introduced by Fix et al. [13,14] where the unique solvability of the variational problem of least-squares type was proved by investigating the assumptions of the Lax–Milgram lemma. This approach was often used for second order problems rewritten as first order systems. For the FEM many authors applied these first order least-squares systems (FOSLS) e.g. to elasticity or Stokes problems (see e.g. [15,11,2,10]).

Bramble and Pasciak [5], or Bramble, Pasciak and Lazarov [7,8] introduced a least-squares functional consisting of a discrete inner product related to the Sobolev space of order −1. This approach was extended to a least-squares method

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involving the coupling of FEM and BEM by Maischak and Stephan [22]. The resulting formulation involves the usage of inner products of the Sobolev spaces $\tilde{H}^{-1}(\Omega)$ and $H^{1/2}(\partial\Omega)$ for second order problems reduced to a system of first order. For the discretization standard finite element and boundary element spaces were used by applying multigrid or BPX. Similarly, Gatica, Harbrecht and Schneider [18] treated exterior boundary value problems where the inner products were realized by wavelet approximations in connection with multilevel preconditioning and multiscale methods. In this paper, we extend the above described least-squares method of Maischak and Stephan to a system of second order applied to a linear elastic transmission problem.

As a model problem we consider a linear elastostatic material for the interior domain Ω and the unbounded region Ω^C. In the interior domain we want to treat also the case of nearly incompressible material. Therefore, we introduce a new variable as Lagrange multiplier. On the interface boundary $\Gamma = \partial\Omega$ we have prescribed jumps for the displacement $\mathbf{u}_0$ and for the traction $\mathbf{t}_0$. The elastic problem in the exterior domain is reduced to a system of strongly elliptic boundary integral equations on Γ. To define our least-squares functional we give a modified representation of the equilibrium equation of the inner problem. The unknowns of the system are the displacement $\mathbf{u}$, the pressure p and the traction $\boldsymbol{\phi}$. We show the existence of an unique solution of the corresponding minimization problem which consists of $\tilde{\mathbf{H}}^{-1}(\Omega)$ and $L^2(\Omega)$ inner products on the domain Ω and of the $\mathbf{H}^{1/2}(\Gamma)$ inner product on the boundary Γ.

For the discretization of the least-squares formulation we use continuous piecewise linear functions for $\mathbf{u}$ and piecewise constant functions for p in Ω and piecewise constant functions for $\boldsymbol{\phi}$ on Γ and discrete versions of the inner products. The discrete least-squares functional consists — in contrast to the continuous formulation — of an additional boundary term in the $\mathbf{H}^{-1/2}(\Gamma)$ inner product. We approximate the inner products by the action of multilevel preconditioners.

The paper is organized as follows. In Section 2 we describe the underlying elasticity transmission problem and reduce the exterior problem in the unbounded domain to boundary integral equations on the boundary Γ. In Section 3 we state the equivalent minimization problem with the least-squares function J. We prove the assertions of the Lax–Milgram lemma and hence get the unique solvability. In the next section we present the discretization of the least-squares formulation with the corresponding discrete bilinear and linear form. Additionally we describe the procedure to deal with the critical inner products. The proof of an a priori error estimate is given in Section 4. With the local error indicators we can steer an adaptive algorithm to get a better mesh refinement and a more effective computation of the solution. Section 5 treat the system of linear equations arising from the discrete variational formulation. Finally, we give in Section 6 some numerical examples to show the efficient use of the least-squares formulation in comparison with the standard Galerkin formulation.

In the following we will write $a \lesssim b$ if $a \leqslant Cb$ with a constant C independent of the mesh size h, and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

2. Transmission problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ and $\Omega^C = \mathbb{R}^d \setminus \bar{\Omega}$ with normal $\mathbf{n}$ on Γ pointing into Ω^C.

For the interior elastic problem in Ω we allow incompressible material. We assume Hooke's law connecting the stress tensor for the displacement $\mathbf{u}$ and the pressure p in form

$$\boldsymbol{\sigma}_{ij}(\mathbf{u}, p) = 2\mu\boldsymbol{\varepsilon}_{ij}(\mathbf{u}) - \delta_{ij} p$$

whereas we assume homogeneous elastic material behavior in Ω^C. Let $\mathbf{F} \in \mathbf{L}^2(\Omega)$, $\mathbf{u}_0 \in \mathbf{H}^{1/2}(\Gamma)$, $\mathbf{t}_0 \in \mathbf{H}^{-1/2}(\Gamma)$, $\lambda \in \mathbb{R}^+$. We consider the model transmission problem of finding $\mathbf{u}_1 \in \mathbf{H}^1(\Omega)$, $\mathbf{u}_2 \in \mathbf{H}^1_{\mathrm{loc}}(\Omega^C)$, $p \in L^2(\Omega)$ such that

$$L(\mathbf{u}_1, p) = \mathbf{F}; \qquad \frac{1}{\lambda} p + \operatorname{div} \mathbf{u}_1 = 0 \qquad \text{in } \Omega, \tag{1}$$

$$-\Delta^* \mathbf{u}_2 = -\hat{\mu}\Delta\mathbf{u}_2 - (\hat{\lambda} + \hat{\mu}) \operatorname{grad} \operatorname{div} \mathbf{u}_2 = 0 \qquad \text{in } \Omega^C, \tag{2}$$

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{u}_0 \qquad \text{on } \Gamma, \tag{3}$$

$$\boldsymbol{\sigma}(\mathbf{u}_1, p) \cdot \mathbf{n} = T_2(\partial., \mathbf{n})\mathbf{u}_2 + \mathbf{t}_0 \qquad \text{on } \Gamma, \tag{4}$$

$$\mathbf{u}_2 = \mathcal{O}\big(|x|^{-1}\big) \qquad \big(|x| \to \infty\big), \tag{5}$$

where $L(\mathbf{u}_1, p) = (L_1(\mathbf{u}_{11}, p), \dots, L_d(\mathbf{u}_{1d}, p))$ with $L_i(\mathbf{u}_{1i}, p) = -\sum_{j=1}^d \frac{\partial \boldsymbol{\sigma}_{ij}}{\partial \mathbf{x}_j} = -(\operatorname{div}\boldsymbol{\sigma})_i$.

The traction $T_2(\partial., \mathbf{n})\mathbf{u}_2$ on the boundary Γ is given by $T_2(\partial., \mathbf{n})\mathbf{u}_2 = \boldsymbol{\sigma}(\mathbf{u}_2) \cdot \mathbf{n} = 2\hat{\mu}\partial_{\mathbf{n}}\mathbf{u}_2 + \hat{\lambda}\mathbf{n} \operatorname{div}\mathbf{u}_2 + \hat{\mu}\mathbf{n} \times \operatorname{curl}\mathbf{u}_2$ and $\mathbf{u}_{23} \equiv 0$ for $d = 2$.

In the following, we will apply the boundary integral equation method in Ω^C and reduce the original problem to a nonlocal transmission problem on the bounded domain Ω.

The fundamental solution of the Lamé operator Δ^* is given by

$$\Gamma_{ij}(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi(d-1)\hat{\mu}(\hat{\lambda} + 2\hat{\mu})} \left((\hat{\lambda} + 3\hat{\mu})\gamma_d(\mathbf{x}, \mathbf{y})\delta_{ij} + (\hat{\lambda} + \hat{\mu}) \frac{(\mathbf{x}_i - \mathbf{y}_i)(\mathbf{x}_j - \mathbf{y}_j)}{|\mathbf{x} - \mathbf{y}|^d} \right)$$