Discrete Mathematics 338 (2015) 1827–1844

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

An optimal bound on the number of moves for open mancala

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ARTICLE INFO

Article history:
Received 30 October 2013
Received in revised form 31 March 2015
Accepted 10 April 2015
Available online 5 June 2015

Keywords:
Discrete-time games
Integer compositions
Zero-player game
Discrete dynamical system

ABSTRACT

We determine the optimal bound for the maximum number of moves required to reach a periodic configuration of *open mancala* (also called open owari), inspired by a popular African game. A mancala move can be interpreted as a map from the set of compositions of a given integer in itself, thus relating our result to the study of the corresponding finite dynamical system.

1. Introduction

Mancala is a family of traditional African games played in many versions and under many names. It comprises a circular list of *holes* containing zero or more *seeds*. A move consists of selecting a nonempty hole, taking all its seeds and *sowing* them in the subsequent holes, one seed per hole.

Here we study an idealized version of this called *open mancala* (see [1], where the game is called *open owari*). We assume there is an infinite sequence of holes. Further we assume that the nonempty holes are consecutive; such a configuration is described by a sequence of positive integers λ_i, $i = 1, \dots, \ell$, giving the number of seeds in each nonempty hole, starting from the leftmost one. We assume a move always selects the leftmost nonempty hole, sowing to the right. Thus the leftmost nonempty hole always advances of one position. It is clear that the sequence of configurations becomes periodic in a finite time. The periodic configurations have been completely classified [1,3,2]: see Section 2.

This game is related to a card game called *Bulgarian Solitaire*, discussed by Gardner [5] in a 1983 *Scientific American* column. Indeed, that game is isomorphic to the case of open mancala where $\lambda_i \geq \lambda_j$ for $i \leq j$ (monotone mancala), see Section 6. Gardner mentioned a conjecture on the maximum number of moves before the onset of periodicity when the number of cards is of the form $k(k+1)/2$. The conjecture was proved independently a few years later by Igusa [7] and Etienne [4]. In 1998, Griggs & Ho [6] revisited the result and proposed a new conjecture in the case of a generic number of cards. Due to the isomorphism between monotone mancala and Bulgarian solitaire, our result is strongly related to this conjecture. However, our results apply to open mancala, not to the monotone variant, hence the conjecture about the Bulgarian Solitaire remains unproven.

The main result of the paper is the optimal bound for the maximum number of moves, before open mancala reaches periodicity, as a function of n, the number of seeds. In Section 3 we prove the lower bound by producing, for any n, a configuration that requires exactly that number of moves. In Section 4 we introduce the main tool in proving the upper

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http://dx.doi.org/10.1016/j.disc.2015.04.013

bound, namely *s-monotonicity*. In Section 5 we prove that every configuration reaches periodicity within that number of moves.

2. Notations and setting

We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}^*$ the set of strictly positive integers. By T_k, $k \in \mathbb{N}$, we denote the kth triangular number

$$T_k = \sum_{i=0}^{k} i = \frac{k(k+1)}{2},$$

where in particular $T_0 = 0$.

Definition 2.1. A *configuration* λ is a sequence of nonnegative numbers λ_i, $i \geq 1$, that is $\lambda : \mathbb{N}^* \to \mathbb{N}$. The *support* $\mathrm{supp}(\lambda)$ of a configuration λ is the set of indices $\{i \in \mathbb{N}^* : \lambda_i > 0\}$.

Definition 2.2. A *mancala configuration* λ is a configuration having a *connected* support of the form $\{1, \dots, \ell\}$ for some $\ell = \mathrm{len}(\lambda) \in \mathbb{N}$. The index $\mathrm{len}(\lambda)$ is called the *length* of the configuration. In the special case of the *zero* configuration, we shall conventionally define its length to be zero. We shall denote by Λ the set of all mancala configurations.

Definition 2.3. If $\lambda \in \Lambda$, its *mass* is the number

$$|\lambda| = \sum_{i=1}^{\infty} \lambda_i = \sum_{i=1}^{\mathrm{len}(\lambda)} \lambda_i.$$

We denote by Λ_n the set of all mancala configurations of mass n.

The *mancala* game, in our setting, is a discrete dynamical system associated with a function $\mathcal{M} : \Lambda \to \Lambda$ mapping the space of *mancala* configurations Λ in itself.

Definition 2.4 (*Sowing*)**.** The *sowing* of a mancala configuration λ is an operation on λ that results in a new configuration $\mu = \mathcal{M}(\lambda)$ defined as follows:

$$\mu_i = \begin{cases} \lambda_{i+1} + 1 & \text{if } 1 \leq i \leq \lambda_1, \\ \lambda_{i+1} & \text{if } i > \lambda_1. \end{cases}$$

Conventionally, $\mathcal{M}$ maps the empty configuration into itself.

It is clear that $\mathcal{M}$ preserves the mass of a configuration, so that it can be restricted to Λ_n.

It is convenient to define the (right) *shift* operator $\mathcal{E}$, that acts on generic sequences simply by a change in the indices.

Definition 2.5 (*Shift Operator*)**.** If $\lambda : \mathbb{N}^* \to \mathbb{N}$ is a sequence, the sequence $\mathcal{E}(\lambda)$ is defined by

$$\mathcal{E}(\lambda)_i = \begin{cases} 0 & \text{if } i = 1, \\ \lambda_{i-1} & \text{if } i > 1. \end{cases}$$

Clearly, the result of the shift operator is never a mancala configuration (with the exception of the empty configuration).

Both $\mathcal{M}$ and $\mathcal{E}$ can be iterated, the symbol $\mathcal{M}^k$ (resp. $\mathcal{E}^k$) denoting the result of k repeated applications of $\mathcal{M}$ (resp. of $\mathcal{E}$). The shift operator has a left inverse $\mathcal{E}^{-1}$, defined by $\mathcal{E}^{-1}(\lambda)_i = \lambda_{i+1}$ and satisfying $\mathcal{E}^{-1}(\mathcal{E}(\lambda)) = \lambda$ for any configuration λ.

Definition 2.6 (*Partial Ordering and Sum*)**.** We say that $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all $i \geq 1$. Moreover we say that $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$. If $\lambda, \mu : \mathbb{N}^* \to \mathbb{Z}$ are two integer sequences (in particular if any of them is a configuration in Λ), we define the sum and difference $\lambda \pm \mu$ componentwise: $(\lambda \pm \mu)_i = \lambda_i \pm \mu_i$.

Remark 2.1 (*Comparison*)**.** It is easy to check that if $\lambda, \mu \in \Lambda$ and $\lambda \leq \mu$, then $\mathcal{M}(\lambda) \leq \mathcal{M}(\mu)$.

Definition 2.7. A configuration is called *monotone* if it is weakly decreasing, i.e. if $\lambda_i \geq \lambda_j$ whenever $i \leq j$.

Remark 2.2 (*Monotone Mancala*)**.** If λ is a monotone configuration, then so it is $\mathcal{M}(\lambda)$. Moreover, if $\lambda \in \Lambda$ has length ℓ, then $\mathcal{M}^{\ell-1}(\lambda)$ is monotone.