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Decompositions of triangle-free 5-regular graphs into paths of length five☆

F. Botler*, G.O. Mota, Y. Wakabayashi

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508-090, São Paulo, Brazil

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ABSTRACT

A P_k-decomposition of a graph G is a set of edge-disjoint paths with k edges that cover the edge set of G. Kotzig (1957) proved that a 3-regular graph admits a P_3-decomposition if and only if it contains a perfect matching. Kotzig also asked what are the necessary and sufficient conditions for a $(2k+1)$-regular graph to admit a decomposition into paths with $2k+1$ edges. We partially answer this question for the case $k = 2$ by proving that the existence of a perfect matching is sufficient for a triangle-free 5-regular graph to admit a P_5-decomposition. This result contributes positively to the conjecture of Favaron et al. (2010) that states that every $(2k+1)$-regular graph with a perfect matching admits a P_{2k+1}-decomposition.

1. Introduction

In this paper, the term decomposition always refer to an edge-decomposition of a graph. Given a graph $G = (V, E)$, a graph decomposition of G is a set of edge-disjoint subgraphs of G that cover E. The problem of finding decompositions of graphs into subgraphs of certain types is a classical problem in graph theory that traces back to the late 19th century. One of the earliest results of this nature is a theorem of Petersen (1891) that states that every $2k$-regular graph can be decomposed into 2-factors. Many surveys and books on this topic have appeared in the literature, among which the reader may refer to [1,4,8,12,17–20].

In general, finding or deciding the existence of some nontrivial graph decomposition is a hard problem, and much effort has been devoted to studying decompositions of particular classes of graphs into some classes of subgraphs. If we restrict our attention to decompositions of arbitrary graphs into cycles or paths, we come across many interesting conjectures and to the following old and elegant result of Lovász [27].

Theorem 1.1 (*Lovász*). *Every n-vertex graph can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles.*

In fact, according to Lovász [27], in 1966 Gallai conjectured that every n-vertex connected graph admits a decomposition into at most $\lceil n/2 \rceil$ paths, and Hajós conjectured that any Eulerian graph can be decomposed into at most $\lfloor n/2 \rfloor$ cycles. We also refer to Bondy [3] for these and other conjectures. Looking for asymptotic results, Erdős and Gallai [13,14] conjectured that every n-vertex graph can be decomposed into $O(n)$ cycles and edges. Many researchers have obtained partial results on these conjectures (see [9,10,15]).

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* Corresponding author.
E-mail addresses: fbotler@ime.usp.br (F. Botler), mota@ime.usp.br (G.O. Mota), yw@ime.usp.br (Y. Wakabayashi).

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Decompositions of regular graphs have been extensively investigated in the last decades, mostly restricted to decompositions into paths of fixed length. We denote by P_k (resp. C_k) a path (resp. cycle) of length k, that is, with k edges. (We observe that this notation is not so standard.) Jacobson, Truszczyński and Tuza [22] proved that every 4-regular bipartite graph admits a P_4-decomposition. For other results concerning $2k$-regular graphs and cartesian products of regular graphs, the reader is referred to [24,29]; and for results on decompositions of regular graphs with large girth, we mention Kouider and Lonc [26].

Kotzig [25] proved that a 3-regular graph G admits a P_3-decomposition if and only if G contains a perfect matching. In fact, Kotzig proved a slightly stronger result (on two P_3-decompositions). The proof used by Kotzig is presented by Bouchet and Fouquet [7]. This result was generalized by Jaeger, Payan, and Kouider [23], who proved that a $(2k+1)$-regular graph that contains a perfect matching can be decomposed into bistars. In another direction, Heinrich, Liu and Yu [21] proved that every $3m$-regular graph without cut-edges admits a P_3-decomposition. Kotzig asked what are the necessary and sufficient conditions for a $(2k+1)$-regular graph G to be decomposable into paths of length $2k+1$. A necessary condition is that G must contain a k-factor. Favaron, Genest, and Kouider [16] proved that this condition is not sufficient. For $k = 2$ (that is, for a 5-regular graph), Favaron, Genest, and Kouider [16] proved that it is sufficient that G contains a perfect matching and no cycles of length four to admit a P_5-decomposition. Here we prove that every triangle-free 5-regular graph that contains a perfect matching admits a P_5-decomposition.

This paper is organized as follows. In Section 2 we give some definitions and establish the notation. In Section 3 we show that triangle-free 5-regular graphs containing a perfect matching admit a decomposition into copies of P_5 and some specific trails T_5 with five vertices. Section 4 contains some lemmas which enable us to reduce the number of copies of T_5 and increase the number of copies of P_5, obtaining a decomposition closer to the desired one. In Section 5 we use the results obtained in Sections 3 and 4 to obtain a P_5-decomposition.

2. Basic definitions and notation

The basic terminology and notation used in this paper are standard (see [2,11]). A *mixed graph* is a simple graph in which some edges may receive an orientation. More precisely, it is a triple $\bar{G} = (V, E, A)$ consisting of a vertex set V, an (undirected) edge set E and a directed edge set A, such that (V, E) is a simple graph and (V, A) is a simple directed graph; and furthermore, no two distinct edges in $E \cup A$ have the same endpoints. Given a mixed graph $\bar{G}$, we denote by $V(\bar{G})$, $E(\bar{G})$ and $A(\bar{G})$, the sets of vertices, undirected edges and directed edges of $\bar{G}$, respectively. Let $\hat{A}(\bar{G})$ be the *underlying edge set* of $A(\bar{G})$, i.e., the set of edges obtained by removing the orientation of the directed edges in $A(\bar{G})$. We denote by G the *underlying graph* of $\bar{G}$, i.e., G is the graph such that $V(G) = V(\bar{G})$ and $E(G) = E(\bar{G}) \cup \hat{A}(\bar{G})$. We note that, in this paper all (mixed) graphs are simple.

Let $\bar{G}$ be a mixed graph. For ease of notation, we write simply ab to refer to an undirected edge $\{a, b\} \in E$ or a directed edge $(a, b) \in A(\bar{G})$, and use the term *edge* to refer to an element that belongs to $E(\bar{G}) \cup A(\bar{G})$. When the orientation of an edge is relevant, we write $ab \in A(\bar{G})$, or specify that ab is a directed edge. A mixed subgraph $\bar{H}$ of $\bar{G}$ is a mixed graph such that $V(\bar{H}) \subseteq V(\bar{G})$, $E(\bar{H}) \subseteq E(\bar{G})$ and $A(\bar{H}) \subseteq A(\bar{G})$. Given a set of mixed subgraphs $\bar{H}_1, \ldots, \bar{H}_k$ of $\bar{G}$, we denote by $\bigcup_{i=1}^{k} \bar{H}_i$ the mixed subgraph $\bar{H} = \left(\bigcup_{i=1}^{k} V(\bar{H}_i), \bigcup_{i=1}^{k} E(\bar{H}_i), \bigcup_{i=1}^{k} A(\bar{H}_i)\right)$.

We say that a mixed graph $\bar{H}$ is a *copy* of a mixed graph $\bar{G}$ if H is isomorphic to G. A *path* P in $\bar{G}$ is a sequence of distinct vertices $P = v_0v_1 \cdots v_k$ such that v_iv_{i+1} is an edge in $\bar{G}$, for $i = 0, 1, \ldots, k-1$. (Note that, possibly $v_{i+1}v_i$ is a directed edge, for some i in $\{0, \ldots, k-1\}$). For convenience, we will also consider that such a path P is a mixed graph with $V(P) = \{v_0, v_1, \ldots, v_k\}$ and $E(P) \cup A(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$. The *length* of P is the number of edges in P. We denote by P_k any path of length k, and we denote by T_k the graph (trail) that is obtained from a path $v_0v_1 \cdots v_{k-1}$ by the addition of the edge $v_{k-1}v_1$. We refer to T_k simply as $v_0v_1 \cdots v_{k-1}v_1$. If a mixed graph $\bar{H}$ is a copy of P_k or T_k we also write $\bar{H} = v_0 \cdots v_k$ or $\bar{H} = v_0 \cdots v_{k-1}v_1$, respectively.

We say that a set $\{\bar{H}_1, \ldots, \bar{H}_k\}$ of mixed graphs is a *decomposition* of a mixed graph $\bar{G}$ if $\bigcup_{i=1}^{k} E(\bar{H}_i) = E(\bar{G})$, $\bigcup_{i=1}^{k} A(\bar{H}_i) = A(\bar{G})$, and furthermore $E(\bar{H}_i) \cap E(\bar{H}_j) = \emptyset$ and $A(\bar{H}_i) \cap A(\bar{H}_j) = \emptyset$ for all $1 \le i < j \le k$. Let $\mathcal{H}$ be a family of graphs. An *$\mathcal{H}$-decomposition* $\mathcal{D}$ of $\bar{G}$ is a decomposition of $\bar{G}$ such that each element of $\mathcal{D}$ is isomorphic to an element of $\mathcal{H}$. If $\mathcal{H} = \{H\}$ we say that $\mathcal{D}$ is an *H-decomposition*.

In the next section we present a result that will allow us to explain the idea behind the proof of the main result, and will also motivate the definitions given thereafter.

3. Canonical $\{P_5, T_5\}$-decomposition

In this section we show that a triangle-free 5-regular graph G that contains a perfect matching is the underlying graph of a mixed graph $\bar{G}$ that admits a $\{P_5, T_5\}$-decomposition that has some special properties. The mixed graph $\bar{G}$ we shall deal with is one obtained from G by assigning an orientation to the edges of each cycle of a given 2-factor F of G, obtaining a set of directed cycles. We shall refer to such an orientation as an *Eulerian orientation* of F. In such a mixed graph, we say that a copy $v_0v_1 \cdots v_5$ of P_5 (resp. a copy $v_0v_1 \cdots v_4v_1$ of T_5) is *canonical* if its directed edges are precisely v_1v_0 and v_4v_5 (resp. v_1v_0 and v_2v_1). If the Eulerian orientation of a 2-factor of G is called $\mathcal{E}$, then we say that a $\{P_5, T_5\}$-decomposition $\mathcal{D}_{\mathcal{E}}$ of $\bar{G}$ is *$\mathcal{E}$-canonical*, or simply *canonical*, if each element of $\mathcal{D}_{\mathcal{E}}$ is canonical.

We will need the following two well-known results.