Discrete Mathematics 338 (2015) 1860–1865

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The complexity of the 3-colorability problem in the absence of a pair of small forbidden induced subgraphs

D.S. Malyshev

National Research University Higher School of Economics, 25/12 Bolshaya Pecherskaya Ulitsa, 603155, Nizhny Novgorod, Russia

ARTICLE INFO

Article history:
Received 20 October 2013
Received in revised form 18 April 2015
Accepted 20 April 2015
Available online 5 June 2015

Keywords:
3-colorability problem
Computational complexity
Polynomial-time algorithm

ABSTRACT

We completely determine the complexity status of the 3-colorability problem for hereditary graph classes defined by two forbidden induced subgraphs with at most five vertices.
© 2015 Elsevier B.V. All rights reserved.

1. Introduction

The coloring problem is one of classical problems on graphs. Its formulation is as follows. A *coloring* is an arbitrary mapping of colors to vertices of some graph. A graph coloring is said to be *proper* if no pair of adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of a graph G is the minimal number of colors in proper colorings of G. The *coloring problem* for a given graph and a number k is to determine whether its chromatic number is at most k or not. The *k-colorability problem* is to verify whether vertices of a given graph can be properly colored with at most k colors.

A graph H is an *induced subgraph* of G if H is obtained from G by deletion of vertices. A *class* is a set of simple unlabeled graphs. A class of graphs is *hereditary* if it is closed under deletion of vertices. It is well known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{Y}$. We write $\mathcal{X} = Free(\mathcal{Y})$ in this case, and the graphs in $\mathcal{X}$ are said to be *$\mathcal{Y}$-free*. If $\mathcal{Y} = \{G\}$, then we will write "G-free" instead of "$\{G\}$-free". If a hereditary class can be defined by a finite set of the forbidden induced subgraphs, then it is said to be *finitely defined*.

The coloring problem for G-free graphs is polynomial-time solvable if G is an induced subgraph of P_4 or $P_3 + K_1$, and it is NP-complete in all other cases [13]. The situation for the k-colorability problem is not clear, even when only one induced subgraph is forbidden. The complexity of the 3-colorability problem is known for all classes of the form $Free(\{G\})$ with $|V(G)| \leq 6$ [4]. A similar result for G-free graphs with $|V(G)| \leq 5$ was recently obtained for the 4-colorability problem [9]. On the other hand, for fixed k, the complexity status of the k-colorability problem is open for P_7-free graphs ($k = 3$), for P_6-free graphs ($k = 4$), and for $P_2 + P_3$-free graphs ($k = 5$).

When we forbid two induced subgraphs, the situation becomes more difficult. For the coloring problem, a complete classification for pairs is open, even if forbidden induced subgraphs have at most four vertices. Although, the complexity is known for some such pairs [8,15,17,18,21]. The same is true for the 3-colorability problem and the five-vertex barrier. We determine here its complexity status for all classes defined by two forbidden induced subgraphs with at most five vertices.

E-mail address: dsmalyshev@rambler.ru.

http://dx.doi.org/10.1016/j.disc.2015.04.019
0012-365X/© 2015 Elsevier B.V. All rights reserved.

2. Notation

For a vertex x of a graph, $deg(x)$ means its degree, $N(x)$ is its neighborhood, $N[x]$ denotes its *closed neighborhood* (i.e. the set $N(x) \cup \{x\}$), $N_k(x)$ is the set of vertices lying at distance k from x. The formula $\Delta(G)$ is the maximum degree of vertices in G.

As usual, P_n, C_n, K_n, O_n, and $K_{p,q}$ stand respectively for the simple path with n vertices, the chordless cycle with n vertices, the complete graph with n vertices, the empty graph with n vertices, and the complete bipartite graph with p vertices in the first part and q vertices in the second. The graph *paw* is obtained from a triangle by adding a vertex and an edge incident to the new vertex and a vertex of the triangle. The graphs *fork*, *bull*, *butterfly* have the vertex set $\{x_1, x_2, x_3, x_4, x_5\}$. The edge set for *fork* is $\{x_1x_2, x_1x_3, x_1x_4, x_4x_5\}$, for *bull* is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_5\}$, for *butterfly* is $\{x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_1x_5, x_4x_5\}$. The graph $hammer_k$ has the vertex set $\{x_1, x_2, x_3, y_1, y_2, \ldots, y_k\}$ and the edges $x_1x_2, x_1x_3, x_2x_3, x_1y_1, y_1y_2, \ldots, y_{k-1}y_k$. Note that $paw = hammer_1$.

The *complement graph* $\overline{G}$ of G is a graph on the same set of vertices, and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *sum* $G_1 + G_2$ is the disjoint union of G_1 and G_2. The disjoint union of k copies of a graph G is denoted by kG. For a graph G and a set $V' \subseteq V(G)$, the formula $G \setminus V'$ denotes the subgraph of G obtained by deleting all vertices in V'.

3. Boundary graph classes

The notion of a boundary graph class is a helpful tool for the analysis of the computational complexity of graph problems in the family of hereditary graph classes. This notion was originally introduced by V.E. Alekseev for the independent set problem [1]. It was later applied for the dominating set problem [3]. A study of boundary graph classes for some graph problems was extended in the paper of Alekseev et al. [2], where the notion was formulated in its most general form. We will give the necessary definitions.

Let Π be an NP-complete graph problem. A hereditary graph class is said to be *Π-easy* if Π is polynomial-time solvable for its graphs. If the problem Π is NP-complete for graphs in a hereditary class, then this class is said to be *Π-hard*. A class of graphs is said to be *Π-limit* if this class is the limit of an infinite monotonically decreasing chain of Π-hard classes. In other words, $\mathcal{X}$ is Π-limit if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \cdots$ of Π-hard classes, such that $\mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k$. A Π-limit class that is minimal under inclusion is said to be *Π-boundary*.

The following theorem certifies the significance of the notion of a boundary class.

Theorem 1 ([1]). *A finitely defined class is Π-hard if and only if it contains some Π-boundary class.*

This theorem shows that knowledge of all Π-boundary classes leads to a complete classification of finitely defined graph classes with respect to the complexity of Π. Two concrete classes of graphs are known to be boundary for several graph problems. The first of them is $\mathcal{S}$. It constitutes all forests with at most three leaves in each connected component. The second one is $\mathcal{T}$, which is the set of line graphs of graphs in $\mathcal{S}$. The paper [2] is a good survey about graph problems, for which either $\mathcal{S}$ or $\mathcal{T}$ is boundary.

Some classes are known to be limit and boundary for the 3-colorability problem. The set $\mathcal{F}$ of all forests and the set $\mathcal{T}'$ of line graphs of forests with degrees at most three are limit classes for it [14]. Some continuum set of boundary classes for the k-colorability problem is known for any fixed $k \geq 3$ [12,19,20].

The main result of this paper can be briefly formulated by means of $\mathcal{F}$ and $\mathcal{T}'$. Namely, if G_1 and G_2 have at most five vertices, then the 3-colorability problem is tractable for $\mathcal{X} = Free(\{G_1, G_2\})$ if $\mathcal{F} \not\subseteq \mathcal{X}$, $\mathcal{T}' \not\subseteq \mathcal{X}$, $\{G_1, G_2\} \neq \{K_{1,4}, bull\}$, $\{G_1, G_2\} \neq \{K_{1,4}, butterfly\}$, and the problem is NP-complete for all other choices of G_1 and G_2 on at most five vertices.

4. NP-completeness of the 3-colorability problem for some graph classes

The results listed above on limit classes for the 3-colorability problem together with Theorem 1 allow us to prove NP-completeness of the problem for some finitely defined classes. Namely, if $\mathcal{Y}$ is a finite set of graphs, and $\mathcal{Y} \cap \mathcal{F} = \emptyset$ or $\mathcal{Y} \cap \mathcal{T}' = \emptyset$, then the problem is NP-complete for $Free(\mathcal{Y})$. But, this idea cannot be applied to $Free(\{K_{1,4}, bull, butterfly\})$, because $K_{1,4} \in \mathcal{F}$, $bull \in \mathcal{T}'$, and $butterfly \in \mathcal{T}'$. Nevertheless, the 3-colorability problem is NP-complete for this class. To show this, we use a graph operation called diamond implantation.

Let G be a graph with a non-leaf vertex x. Applying a *diamond implantation* to x implies:

- an arbitrary splitting $N(x)$ into two nonempty parts A and B
- deletion of x and addition of new vertices y_1, y_2, y_3, y_4
- addition of all edges of the form y_1a, $a \in A$ and of the form y_4b, $b \in B$
- addition of the edges $y_1y_2, y_1y_3, y_2y_3, y_2y_4, y_3y_4$

Clearly, for every graph G and every non-leaf vertex in G, applying the diamond implantation preserves 3-colorability. This property and the paper [11] give the key idea of the proof of Lemma 1.

Download English Version:

https://daneshyari.com/en/article/6423289

Download Persian Version:

https://daneshyari.com/article/6423289

Daneshyari.com