Discrete Mathematics 338 (2015) 1883–1906

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The formulas for the number of spanning trees in circulant graphs ☆

Min Li [a], Zhibing Chen [a], Xiaoqing Ruan [a], Xuerong Yong [b,*]

[a] College of Mathematics and Computational Science, Shenzhen University, Shenzhen, China

[b] Department of Mathematical Sciences, University of Puerto Rico, Mayaguez, USA

ARTICLE INFO

Article history:
Received 2 July 2014
Received in revised form 25 April 2015
Accepted 27 April 2015
Available online 5 June 2015

Keywords:
Circulant graphs
Spanning trees
Asymptotic limit
Chebyshev polynomials

ABSTRACT

The circulant graph $C_n^{s_1,s_2,\ldots,s_t}$ is the $2t$ regular graph with n vertices labeled $0, 1, 2, \ldots, n-1$, where each vertex i has the $2t$ neighbors $i \pm s_1, i \pm s_2, \ldots, i \pm s_t$ in which all the operations are modulo n. Golin et al. (2010) derive several closed integral formulas for the asymptotic limit

$$\lim_{n\to\infty} T\left(C_n^{s_1,s_2,\ldots,s_k,\lfloor \frac{n}{d_1}\rfloor+e_1,\lfloor \frac{n}{d_2}\rfloor+e_2,\ldots,\lfloor \frac{n}{d_l}\rfloor+e_l}\right)^{\frac{1}{n}},$$

as a function of s_i, d_j and e_k, where $T(G)$ is the number of spanning trees in graph G.

In this paper we derive simple and explicit formulas for the number of spanning trees in circulant graphs $C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}$. Following from the formulas we show that

$$\lim_{n\to\infty} T\left(C_{pn}^{1,a_1n,a_2n,\ldots,a_ln}\right)^{\frac{1}{n}} = \prod_{t=0}^{k-1}\left(\sqrt{1+\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}}+\sqrt{\sum_{i=1}^{l}\sin^2\frac{\pi a_i t}{p}}\right)^{\frac{2p}{k}},$$

where $k = \operatorname{lcm}(\frac{p}{a_1}, \frac{p}{a_2}, \ldots, \frac{p}{a_l})$, and lcm denotes the least common multiple. The asymptotic limit represents the average growth rate of the number of spanning trees.

The research is continuation of the previous work (Golin et al., 2010; Zhang et al., 2000; Zhang et al., 2005).

Published by Elsevier B.V.

1. Introduction

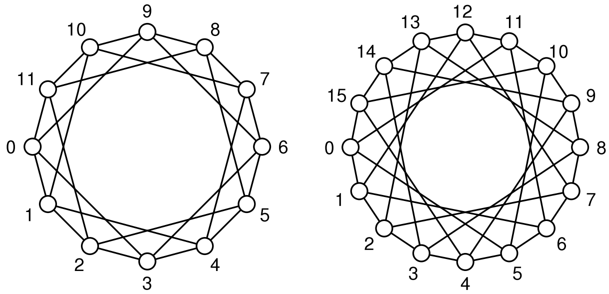
Throughout this paper, the graphs are allowed to contain multiple edges and self-loops unless otherwise specified.

Let G be a connected graph on n vertices. *A spanning tree* in G is a tree having the same vertex set as G and its edge set is a subset of the edge set of G. *An oriented spanning tree* in a directed graph $\vec{G}$ is a rooted tree with the same vertex set as $\vec{G}$,

☆ The research was supported in part by Shenzhen University. The fourth author's work was also supported by DIMACS and the University of Puerto Rico at Mayaguez.

* Corresponding author.
E-mail addresses: lminyy@sina.com (M. Li), chenzb@szu.edu.cn (Z. Chen), ruanxq@szu.edu.cn (X. Ruan), xryong@dimacs.rutgers.edu (X. Yong).

http://dx.doi.org/10.1016/j.disc.2015.04.025
0012-365X/Published by Elsevier B.V.



(a) $C_{12}^{1,3}$. (b) $C_{16}^{1,5}$.

Fig. 1. Two special circulant graphs.

i.e., there is a specified root node and paths from it to every vertex of $\vec{G}$. The number of spanning trees has a long research history because it is interesting from a combinatorial perspective and also arises in several application problems. An old motivation is that the number of spanning trees characterizes basically the reliability of a network in the presence of line fault and counting the number of spanning trees is important in designing electrical circuits [7]. Another related discovery is that the resistance distance between two nodes in a network can be expressed by the number of spanning trees and therefore computation of the effective electric resistance involves calculation/estimation of the number of spanning trees in a given network (see e.g., Theorem 7–4 in [16], or [1,17]).

Given the adjacency matrix of a graph G, *Kirchoff's matrix tree theorem* [11] gives a closed formula for calculating the number of spanning trees. However, for a given general graph it is not easy to count the exact number (or to get a good estimation) from the theorem and therefore special graphs have been considered extensively because it turns out to be possible to derive explicit formulas for them. The real problem, then, is to calculate the number of spanning trees of special graphs in particular parameterized classes. The class of circulant graphs to be considered here is one of the classes of graphs that have been received much attention in the last decades.

Circulant graphs can be defined in different ways [2,4,10]. Here it is convenient to introduce them and the values to be counted as described in [10]. Let $s_1 < s_2 < \cdots < s_k \le \lfloor \frac{n}{2} \rfloor$ be given positive integers. The circulant graph on n vertices and jumps $s_1, s_2, \ldots, s_k$ is defined by

$$C_n^{s_1,s_2,\ldots,s_k} = (V, E)$$

where $V = \{0, 1, 2, \ldots, n-1\}$, and

$$E = \bigcup_{i=0}^{n-1} \{ (i, i+s_1),\ (i, i+s_2), \ldots, (i, i+s_k)\}$$ [1]

where all of the additions are done $\mod n$. That is, each node is connected to the nodes that are jumps s_j away from it, for $j = 1, 2, \ldots, k$. Similarly, the directed circulant graph on n vertices,[2] $\vec{C}_n^{s_1,s_2,\ldots,s_k}$, has the same vertex set but $s_k < n$ and

$$E = \bigcup_{i=0}^{n-1} \{ (i, i+s_1),\ (i, i+s_2), \ldots, (i, i+s_k)\}$$

where an edge is *directed* from each i to the nodes s_j ahead of it, for $j = 1, 2, \ldots, k$. To avoid confusion, we emphasize that, since we are allowing multiple edges in our graphs, $C_n^{s_1,s_2,\ldots,s_k}$ is *always* $2k$-regular and $\vec{C}_n^{s_1,s_2,\ldots,s_k}$ is *always* k-regular. For example, in our notation, $C_{2n}^{1,n}$ is the 4-regular graph with $2n$ vertices such that each vertex i is connected by *one* edge to each of $(i-1) \bmod 2n$ and $(i+1) \bmod 2n$ and by *two* edges to $(i+n) \bmod 2n$. Our techniques would, with slight technical modifications, also permit analyzing graphs in which multiple edges are *not* allowed, e.g., the *Mobius ladder* M_{2n}. This is the 3 regular graph with $2n$ vertices such that each vertex i is connected by one edge to each of $(i-1) \bmod 2n$, $(i+1) \bmod 2n$ and $(i+n) \bmod 2n$. The reason that we do not explicitly analyze such graphs is that such an analysis would require rewriting all of our theorems a second time to deal with these special instances without introducing any new interesting techniques. Two undirected circulant graphs are given in Fig. 1.

Let $T(X)$ stand for the number of spanning trees in a (directed or undirected) graph X. For any fixed integers $1 \le s_1 \le s_2 \le \cdots \le s_k$, it was shown in [19] that, in the case of *directed circulant graphs*,

$$\lim_{n\to\infty} T(\vec{C}_n^{s_1,s_2,\ldots,s_k})^{\frac{1}{n}} = \lim_{n\to\infty} \frac{T(\vec{C}_{n+1}^{s_1,s_2,\ldots,s_k})}{T(\vec{C}_n^{s_1,s_2,\ldots,s_k})} = k,$$

[1] We should mention that the general concept of a set does not allow for multiple copies of an element. Here multiple edges can appear in the graph when performing addition modulo n on the stage of interpreting the elements of E.

[2] In this paper we do not discuss directed graphs. We talk about them for completeness and the reason we address some of their results is that there is an open problem on them, which is emphasized in the Concluding Remarks.

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