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On the interval chromatic number of proper interval graphs

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ABSTRACT

A perfect graph is a graph every subgraph of which has a chromatic number equal to its clique number (Berge, 1963; Lovász, 1972). A (vertex) weighted graph is a graph with a weight function w on its vertices. An interval coloring of a weighted graph maps each vertex v to an interval of size $w(v)$ such that the intervals corresponding to adjacent vertices do not intersect. The size of a coloring is the total size of the union of these intervals. The minimum possible size of an interval coloring of a given weighted graph is its interval chromatic number. The clique number of a weighted graph is the maximum weight of a clique in it. Clearly, the interval chromatic number of a weighted graph is at least its clique number. A graph is superperfect if for every weight function on its vertices, the chromatic number of the weighted graph is equal to its clique number (Hoffman, 1974). It is known that determining the interval chromatic number of a given interval graph is NP-Complete, implying that interval graphs are not included in the family of superperfect graphs (Golumbic, 2004). The question whether these results hold for simple graph families is open since then. We answer this question affirmatively for proper interval graphs for which most investigated problems are polynomial (http://www.graphclasses.org/classes/gc_298.html). Specifically, we show that determining the interval chromatic number of a proper interval graph is NP-Complete in the strong sense. We present a simple 2-approximation algorithm for this special case, whereas the best known approximation algorithm for interval graphs is a $(2+\epsilon)$-approximation (Buchsbaum et al., 2004).

1. Introduction

1.1. Background

The notion of *perfect graphs* was introduced by Berge in the early 1960s [2]. A graph is perfect if every induced subgraph of it satisfies two properties: (a) its chromatic number is equal to its clique number, and (b) its clique cover number is equal to its independence number. Berge presented the famous perfect graph conjecture that was later proven by Lovász [16], according to which these two properties are equivalent. Therefore, we use only the first property and say that a graph is perfect if every induced subgraph of it has a chromatic number equal to its clique number. Several important graph families are included in the family of perfect graphs. A *chord* of a cycle C in a graph is an edge of the graph connecting two vertices that are non-adjacent in C. A graph is *chordal* (or *triangulated*) if every cycle of it with at least 4 vertices has a chord. Chordal graphs are perfect [12,1]. An *interval graph* is a graph whose vertices correspond to open intervals on the real line and two vertices are adjacent if their corresponding intervals intersect. It is easy to see that interval graphs are chordal, therefore

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perfect. A *proper interval* graph is an interval graph where none of the intervals corresponding to the vertices is properly contained in another.

The notions of coloring and perfect graphs are extended as follows. Consider a graph with a weight function w on its vertices. An *interval coloring* c of a weighed graph maps each vertex v to an interval $c(v)$ of size $w(v)$ such that $c(u)$ and $c(v)$ do not intersect whenever u and v are adjacent in the graph. Interval coloring extends the classical notion of vertex coloring. The size of an interval coloring is the size of the union of the intervals. The *interval chromatic number* of a weighted graph is the minimum possible size of an interval coloring of it. The *clique number* of a weighted graph is the maximum total weight of a clique of it. The interval chromatic number of a weighted graph is at least its clique number. Indeed, given a clique K of weight equal to the clique number of the weighted graph, the vertices of K have to be assigned pairwise disjoint intervals. A graph is superperfect if for every weight function w, the chromatic number of the corresponding weighted graph is equal to its clique number [14].

1.2. Related work

The problem of determining the interval chromatic number of an interval graph is known also as the *shipbuilding* problem (see [10]). It is also mentioned in the literature as the *dynamic storage allocation* problem. (see problem (SR2) in [6]). The problem can be described as coloring of *floating* rectangles that are free to move along one axis. Another rectangle coloring variant discussed in the literature is the *berth allocation problem*, in which the rectangles can move along both axes, though not completely freely [18].

Many problems that are NP-Hard in general graphs, such as maximum independent set, maximum clique, chromatic number, can be solved in polynomial time for interval graphs. In particular, for proper interval graphs most investigated problems are polynomial [11]. Larry Stockmeyer showed in 1976 that determining the interval chromatic number of a weighted interval graph is NP-Complete. In fact he showed that the question whether the interval chromatic number of a weighted interval graph is equal to its clique number poses an NP-Complete problem (for a proof see [3]). This implies that some interval graphs are not superperfect, because the problem of determining the clique number of a weighted interval graph is clearly polynomial.

A 6-approximation algorithm for the problem of determining the interval chromatic number of a weighted interval graph was presented in [15]. In [7] and [8] this ratio was improved to 5 and 4, respectively. The best approximation ratio for this problem is $(2 + \epsilon)$ [4].

To develop our positive results we use tools introduced in [17], that investigates a related problem.

It was noted by Alan Hoffman that comparability graphs are superperfect. An infinite family of non-comparability superperfect graphs is demonstrated in [9]. In [5] a forbidden subgraph characterization of comparability split graphs is provided. Since these forbidden subgraphs are not superperfect and comparability graphs are superperfect, this implies a characterization of superperfect graphs within the family of split graphs.

1.3. Our contribution

In this work we consider proper interval graphs. In Section 2 we introduce definitions and notation with some preliminary results. In Section 3 we show that determining whether the interval chromatic number of a given weighted proper interval graph is equal to its clique number is NP-Complete in the strong sense. This implies that the family of proper interval graphs is not included in the family of superperfect graphs, and an infinite family of non-superperfect proper interval graphs can be constructed from our proofs. In Section 4 we initiate the study of approximation algorithms for the interval coloring problem of proper interval graphs by presenting a simple 2-approximation algorithm for it. We conclude with a summary and open questions in Section 5.

2. Preliminaries

Interval Notation: We denote an open (resp. closed) interval I on the real line as $(s(I), t(I))$ (resp. $[s(I), t(I)]$) where $s(I) < t(I)$. For two intervals I, I', we denote by $I < I'$ the fact that $t(I) \leq s(I')$. For a number x and an interval (s, t) we define $x \cdot (s, t) \stackrel{def}{=} (x \cdot s, x \cdot t)$, and $x + (s, t) \stackrel{def}{=} (x + s, x + t)$. We extend these definitions to sets of intervals and functions of intervals as expected. Namely, given a set $\mathcal{I}$ of intervals and a function $f : D \to \mathcal{I}$ over some domain D and a number x we denote $x \cdot \mathcal{I} \stackrel{def}{=} \{x \cdot I : I \in \mathcal{I}\}$, $x + \mathcal{I} \stackrel{def}{=} \{x + I : I \in \mathcal{I}\}$, $(x \cdot f)(d) = x \cdot f(d)$ and $(x + f)(d) = x + f(d)$ for every $d \in D$. The size $size(I)$ of an interval is $t(I) - s(I)$. The size $size(\mathcal{I})$ of a set of pairwise disjoint intervals $\mathcal{I}$ is $size(\cup \mathcal{I}) \stackrel{def}{=} \sum_{I \in \mathcal{I}} size(I)$.

Superperfect Graphs: A *clique* of a graph $G = (V(G), E(G))$ is a subset $K \subseteq V(G)$ of vertices that are pairwise adjacent in G. A clique is *maximal* if it is not contained in any other clique, and it is *maximum* if it contains the biggest number of vertices among all cliques. A *clique cover* of G is a set $\mathcal{K}$ of cliques of G that covers all the vertices, i.e. $\bigcup \mathcal{K} = V(G)$. The *clique number* of G is the size of its maximum cliques and is denoted by $\omega(G)$. A stable (or independent) set of a graph is a subset of its vertices that are pairwise non-adjacent. A vertex is called *simplicial* if all its neighbors constitute a clique. In other words, a vertex is simplicial if it is contained in exactly one maximal clique. A *coloring* of a graph G is a labeling of its vertices, such