Discrete Mathematics 338 (2015) 1928–1937

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the growth of Stanley sequences

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ARTICLE INFO

Article history:
Received 30 September 2014
Received in revised form 2 April 2015
Accepted 3 April 2015
Available online 6 June 2015

Keywords:
Stanley sequence
3-free set
Arithmetic progression
Roth's theorem
Greedy algorithm

ABSTRACT

From an initial list of nonnegative integers, we form a *Stanley sequence* by recursively adding the smallest integer such that the list remains increasing and no three elements form an arithmetic progression. Odlyzko and Stanley conjectured that every Stanley sequence (a_n) satisfies one of two patterns of asymptotic growth, with no intermediate behavior possible. Sequences of Type 1 satisfy $\alpha/2 \le \liminf_{n\to\infty} a_n/n^{\log_2 3} \le \limsup_{n\to\infty} a_n/n^{\log_2 3} \le \alpha$, for some constant α, while those of Type 2 satisfy $a_n = \Theta(n^2/\log n)$. In this paper, we consider the possible values for α in the growth of Type 1 Stanley sequences. Whereas Odlyzko and Stanley considered only those Type 1 sequences for which α equals 1, we show that α can in fact be any rational number that is at least 1 and for which the denominator, in lowest terms, is a power of 3.

1. Introduction

Since the work of Paul Erdős, discrete mathematicians have recognized that the behavior of random objects may be predictable and interesting. In graph theory, for example, Erdős–Rényi random graphs satisfy many properties that are extremely difficult to construct deterministically. Conversely, other properties are not satisfied by random objects, but may appear when a specific structure is imposed. Stanley sequences straddle the line between randomness and determinism, and have largely remained a mystery since their discovery in 1978. While most examples are disorderly, a select few admit beautifully succinct descriptions. Strikingly, the two types appear to follow two very different types of asymptotic growth, with no intermediate behavior possible; however, a proof of this dichotomy has remained elusive. In this paper, we show how the asymptotic growth rate of a "well-structured" Stanley sequence can fall anywhere on a relatively broad spectrum.

A set is *3-free* if no three elements form an arithmetic progression. Odlyzko and Stanley [4] introduced the natural idea of constructing 3-free sets by the greedy algorithm, starting with some finite set of elements. Specifically, let A be a 3-free set of nonnegative integers $\{a_0, a_1, \dots, a_k\}$ satisfying $0 = a_0 < a_1 < \cdots < a_k$. The *Stanley sequence* $S(A)$ is the infinite sequence (a_n) of nonnegative integers defined greedily such that the 3-free property is preserved. That is, for $n > k$, we pick $a_n > a_{n-1}$ to be the smallest integer for which the set $\{a_0, a_1, \dots, a_n\}$ is 3-free. For simplicity we will often denote $S(\{a_0, a_1, \dots, a_k\})$ by $S(a_0, a_1, \dots, a_k)$.

The simplest Stanley sequence is $S(0)$, which begins $0, 1, 3, 4, 9, 10, 12, 13, 27, \dots$. It is easy to show that the nth term of this sequence is the number obtained by writing n in binary and interpreting it in ternary. In particular, the term a_{2^k} equals 3^k. Odlyzko and Stanley [4] found equally explicit expressions, involving ternary digits, for $S(0, 3^n)$ and $S(0, 2\cdot 3^n)$, again finding that the term a_{2^k} equals 3^k for large enough k.

Odlyzko and Stanley observed that some Stanley sequences, such as $S(0)$, have a regular structure and that their asymptotic behavior resembles $a_{2^k} = 3^k$, while all other Stanley sequences are more disorderly and grow at a faster rate. The conjecture is never stated formally in [4]; we phrase it as follows:

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http://dx.doi.org/10.1016/j.disc.2015.04.006
0012-365X/

Conjecture 1.1 (*Based on Work by Odlyzko and Stanley*)**.** *Every Stanley sequence (a_n) follows one of two types of asymptotic growth.*

Type 1*: $\alpha/2 \leq \liminf_{n\to\infty} a_n/n^{\log_2 3} \leq \limsup_{n\to\infty} a_n/n^{\log_2 3} \leq \alpha$, where α is a constant, or*
Type 2*: $a_n = \Theta(n^2/\log n)$.*

Odlyzko and Stanley [4] observed Type 1 behavior only in the case of α equal to 1, for which the sequences $S(0)$, $S(0, 3^n)$, and $S(0, 2 \cdot 3^n)$ are all examples (see Proposition 2.1 and Corollary 2.3). Erdős et al. [1] later found that the sequence $S(0, 1, 4)$ satisfies $a_{2^k} = 3^k + 2^{k-1}$ (for $k \geq 2$) and is of Type 1 with $\alpha = 1$. However, Rolnick [5] demonstrated that many Stanley sequences follow Type 1 growth for other values of α. One example is the sequence $S(0, 1, 7)$, for which we have $a_{2^k} = (10/9) \cdot 3^k$ and $\alpha = 10/9$. Given a Type 1 sequence, we refer to α as a *scaling factor* for the sequence. For all known Type 1 Stanley sequences, the scaling factor is unique.

To date, no Stanley sequence has been proven conclusively to follow Type 2 growth, even though it is believed that almost all Stanley sequences are of this form. Empirical observations by Lindhurst [2] suggest that the sequence $S(0, 4)$ is indeed of Type 2; however it remains possible that the behavior changes suddenly and unexpectedly after a million terms. A probabilistic argument by Odlyzko and Stanley [4] considered a "random" Stanley sequence defined in terms of probability distributions, and showed that such a "sequence" follows Type 2 growth, but does not prove that any actual Stanley sequence is of this form.

In a recent paper, Moy [3] solved a problem posed by Erdős et al. [1], showing that every Stanley sequence (a_n) satisfies $a_n \leq n^2/(2+\epsilon)$ for large enough n. Another problem of [1] remains open, that of finding a Stanley sequence (a_n) satisfying $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$. However, a related question of [1] was resolved by Savchev and Chen [8], who constructed a sequence (a_n) (not a Stanley sequence) satisfying $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$ and such that (a_n) defines a *maximal* 3-free set, that is, a 3-free set that is not a proper subset of any other 3-free set.

In this paper, we consider which growth rates are possible for Type 1 Stanley sequences. Results by Rolnick [5] imply that scaling factors of Type 1 Stanley sequences may be arbitrarily high. Here we prove a much stronger result, given in Theorem 2.5. Let α be a rational number at least 1 and for which the denominator is a power of 3. Then, there exists a Type 1 Stanley sequence with α as a scaling factor. We also consider the *repeat factor* of certain Type 1 sequences. Informally, the repeat factor is the integer a_n at which the sequence begins to exhibit its asymptotic pattern of behavior; a formal definition is given in the next section. We demonstrate that every sufficiently large integer is the repeat factor of some Type 1 Stanley sequence.

2. Preliminaries

Some preliminary definitions and results are required before we can state our main result, Theorem 2.5. We begin by verifying that the simplest Stanley sequence, $S(0)$, does indeed follow Type 1 growth. We will use this fact to prove that many other Stanley sequences also follow Type 1 growth.

Proposition 2.1. *The sequence $S(0)$ follows Type 1 growth with 1 as its unique scaling factor.*

Proof. Let (s_n) denote the sequence $S(0)$. We will prove a slightly stronger result than Type 1 growth; we claim that, for each n, we have

$$1/2 \leq s_n/n^{\log_2 3} \leq 1.$$

We begin by writing n in binary: $n = 2^{d_1} + \cdots + 2^{d_k}$, where we have $d_1 > \cdots > d_k > 0$. We have already noted that s_n equals $3^{d_1} + \cdots + 3^{d_k}$. We conclude

$$\frac{s_n}{n^{\log_2 3}} = \frac{3^{d_1} + \cdots + 3^{d_k}}{\left(2^{d_1} + \cdots + 2^{d_k}\right)^{\log_2 3}} =: f(d_1, \ldots, d_k).$$

Observe that we have

$$(2^{d_1} + \cdots + 2^{d_k})^{\log_2 3} \geq 3^{d_1} + \cdots + 3^{d_k},$$

from which we conclude: $s_n/n^{\log_2 3} \leq 1$.

Now, we compute:

$$\frac{\partial f}{\partial d_k} = \frac{(\ln 3)\left(3^{d_k}\right) - (\ln 2)(\log_2 3)\left(2^{d_k}\right)\left(3^{d_1} + \cdots + 3^{d_k}\right)\left(2^{d_1} + \cdots + 2^{d_k}\right)^{-1}}{\left(2^{d_1} + \cdots + 2^{d_k}\right)^{\log_2 3}}.$$

Observe that we have $(\ln 2)(\log_2 3) = \ln 3$. Hence, the numerator is negative under the following condition:

$$\frac{3^{d_k}}{2^{d_k}} < \frac{3^{d_1} + \cdots + 3^{d_k}}{2^{d_1} + \cdots + 2^{d_k}}.$$