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New bounds for chromatic polynomials and chromatic roots

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ABSTRACT

If G is a k-chromatic graph of order n then it is known that the chromatic polynomial of G, $\pi(G,x)$, is at most $x(x-1)\cdots(x-(k-1))x^{n-k} = (x)_{\downarrow k}x^{n-k}$ for every $x \in \mathbb{N}$. We improve here this bound by showing that

$$\pi(G,x) \le (x)_{\downarrow k}(x-1)^{\Delta(G)-k+1}x^{n-1-\Delta(G)}$$

for every $x \in \mathbb{N}$, where $\Delta(G)$ is the maximum degree of G. Secondly, we show that if G is a connected k-chromatic graph of order n where $k \ge 4$ then $\pi(G,x)$ is at most $(x)_{\downarrow k}(x-1)^{n-k}$ for every real $x \ge n-2+\left(\binom{n}{2}-\binom{k}{2}-n+k\right)^2$ (it had been previously conjectured that this inequality holds for all $x \ge k$). Finally, we provide an upper bound on the moduli of the chromatic roots that is an improvement over known bounds for dense graphs.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ (the *order* and *size* of the graph are, respectively, $|V(G)|$ and $|E(G)|$). For a nonnegative integer x, an *x-colouring* of G is a function $f: V(G) \to \{1,\ldots,x\}$ such that $f(u) \ne f(v)$ for every $uv \in E(G)$. The *chromatic number* $\chi(G)$ is smallest x for which G has an x-colouring. We say that G is *k-chromatic* if $\chi(G)=k$. The well known *chromatic polynomial* $\pi(G,x)$ is the polynomial whose values at nonnegative integral values of x count the number of x-colourings of G. The fact that $\pi(G,x)$ is a polynomial in x follows from the well-known *edge addition–contraction formula*:

$$\pi(G,x) = \pi(G+uv,x) + \pi(G\cdot uv,x)$$

if u and v are nonadjacent vertices of G. An *i-colour partition* of G is a partition of the vertices of G into i nonempty independent sets. Let $a_i(G)$ denote the number of i-colour partitions of G. It is easy to see that

$$\pi(G,x) = \sum_{i=\chi(G)}^{n} a_i(G)\,(x)_{\downarrow i}$$

where $(x)_{\downarrow i} = x(x-1)\ldots(x-i+1)$ is the ith *falling factorial* of x and n is the order of G. Moreover, $a_i(G)$ also satisfies an edge addition–contraction formula, namely, $a_i(G) = a_i(G+uv) + a_i(G\cdot uv)$. We refer the reader to [1] for a general discussion of graph colourings and chromatic polynomials.

Let $\mathcal{G}_k(n)$ be the family of all k-chromatic graphs of order n. Given a natural number $x \ge k$, it is natural to enquire about the maximum number of x-colourings among k-chromatic graphs of order n, that is, among graphs in $\mathcal{G}_k(n)$. Tomescu [7] studied this problem and showed the following:

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Theorem 1.1 ([7, pg. 239]). *Let G be a graph in $\mathcal{G}_k(n)$. Then for every $x \in \mathbb{N}$,*

$$\pi(G, x) \leq (x)_{\downarrow k}\, x^{n-k}.$$

Moreover, when $x \geq k$, the equality is achieved if and only if $G \cong K_k \cup (n-k)K_1$ (the graph consisting of a k-clique plus $n-k$ isolated vertices).

The next natural problem is to maximize the number of x-colourings of a graph over the family of *connected* k-chromatic graphs of order n (we denote this family by $\mathcal{C}_k(n)$). Interestingly, the problem becomes much more complicated when the connectedness condition is imposed. The answer is trivial when $x = k = 2$, as any 2-chromatic connected graph has precisely two 2-colourings. It is well known that (see, for example, [1]) if G is a connected graph of order n then $\pi(G, x) \leq x(x-1)^{n-1}$ for every $x \in \mathbb{N}$ and furthermore, when $x \geq 3$ the equality is achieved if and only if G is a tree. Therefore, for $k = 2$ and $x \geq 3$, the maximum number of x-colourings of a graph in $\mathcal{C}_2(n)$ is equal to $x(x-1)^{n-1}$ and extremal graphs are trees.

Tomescu settled the problem for $x = k = 3$ in [6] and later extended it for $x \geq k = 3$ in [9] by showing that if G is a graph in $\mathcal{C}_3(n)$ then

$$\pi(G, x) \leq (x-1)^n - (x-1) \quad \text{for odd } n$$

and

$$\pi(G, x) \leq (x-1)^n - (x-1)^2 \quad \text{for even } n$$

for every integer $x \geq 3$ and furthermore the extremal graph is the odd cycle C_n when n is odd and odd cycle with a vertex of degree 1 attached to the cycle (denoted C_{n-1}^1) when n is even.

One might subsequently think that maximizing the number of x-colourings of a graph in $\mathcal{C}_k(n)$ should depend on the value of k. Let $\mathcal{C}_k^*(n)$ be the set of all graphs in $\mathcal{C}_k(n)$ which have size $\binom{k}{2} + n - k$ and clique number k (that is, $\mathcal{C}_k^*(n)$ consists of graphs which are obtained from a k-clique by recursively attaching leaves). In [5] Tomescu considered the problem for $x = k \geq 4$ and conjectured the following (see also [8,9]):

Conjecture 1.2 ([5]). *Let G be a graph in $\mathcal{C}_k(n)$ where $k \geq 4$. Then*

$$\pi(G, k) \leq k!\,(k-1)^{n-k},$$

or, equivalently, $a_k(G) \leq (k-1)^{n-k}$, with the extremal graphs belong to $\mathcal{C}_k^*(n)$.

The authors in [1] mention the following conjecture which broadly extends Conjecture 1.2 to all nonnegative integers x:

Conjecture 1.3 ([1, pg. 315]). *Let G be a graph in $\mathcal{C}_k(n)$ where $k \geq 4$. Then for every $x \in \mathbb{N}$,*

$$\pi(G, x) \leq (x)_{\downarrow k}(x-1)^{n-k}.$$

Moreover, for $x \geq k$, the equality holds if and only if G belongs to $\mathcal{C}_k^*(n)$.

It is not hard to see that Conjecture 1.3 implies Theorem 1.1 because the chromatic polynomial of a graph is equal to the product of chromatic polynomials of its connected components. However, the problem of maximizing the number of colourings appears more difficult when graphs are connected, since the answer to this problem depends on the value of k (the structure of extremal graphs seem to be different for $k = 2$ and 3). As Tomescu points out [7], the difficulty may lie in the lack of a characterization of *k-critical* graphs (those minimal with respect to k-chromaticity) when $k \geq 4$.

If $G \in \mathcal{C}_k^*(n)$ then it is known that (see, for example, [9]) $\pi(G, x) = (x)_{\downarrow k}\,(x-1)^{n-k}$ as one can first colour the clique of order k and then recursively colour the remaining vertices (which have only one coloured neighbour). On the other hand, one can see that if $\pi(G, x) = (x)_{\downarrow k}\,(x-1)^{n-k}$ then $G \in \mathcal{C}_k^*(n)$ because the multiplicity of the root 1 of the chromatic polynomial of a graph G is equal to the number of blocks of G [1, pg. 35] (a *block* of G is a maximal connected subgraph of G that has no cut-vertex). Therefore, in Conjecture 1.3, the extremal graphs are automatically determined if one can show that $\pi(G, x) \leq (x)_{\downarrow k}\,(x-1)^{n-k}$.

In this article, we first improve Tomescu's general upper bound (Theorem 1.1), and show that if $G \in \mathcal{G}_k(n)$, then

$$\pi(G, x) \leq (x)_{\downarrow k}(x-1)^{\Delta(G)-k+1} x^{n-1-\Delta(G)}$$

for every $x \in \mathbb{N}$ (Theorem 2.2). Secondly, we discuss Conjecture 1.3 and show that if $G \in \mathcal{C}_k(n)$ where $k \geq 4$ then $\pi(G, x)$ is at most $(x)_{\downarrow k}(x-1)^{n-k}$ for every real $x \geq n - 2 + \left(\binom{n}{2} - \binom{k}{2} - n + k\right)^2$ (Theorem 2.5). Finally, we also give a new upper bound on the moduli of the chromatic roots of a graph (Theorem 2.7); our bound improves previously known bounds for dense graphs.