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Some bounds on the neighbor-distinguishing index of graphs

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ABSTRACT

A proper edge coloring of a graph G is neighbor-distinguishing if any two adjacent vertices have distinct sets consisting of colors of their incident edges. The neighbor-distinguishing index of G is the minimum number $\chi'_a(G)$ of colors in a neighbor-distinguishing edge coloring of G.

Let G be a graph with maximum degree Δ and without isolated edges. In this paper, we prove that $\chi'_a(G) \leq 2\Delta$ if $4 \leq \Delta \leq 5$, and $\chi'_a(G) \leq 2.5\Delta$ if $\Delta \geq 6$. This improves a result in Zhang et al. (2014), which states that $\chi'_a(G) \leq 2.5\Delta + 5$ for any graph G without isolated edges. Moreover, we prove that if G is a semi-regular graph (i.e., each edge of G is incident to at least one Δ-vertex), then $\chi'_a(G) \leq \frac{5}{3}\Delta + \frac{13}{3}$.

1. Introduction

All graphs considered in this paper are finite and simple. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. Let $N_G(v)$ denote the set of neighbors of a vertex v in G and $d_G(v) = |N_G(v)|$ denote the degree of v in G. The vertex v is called a *k-vertex* if $d_G(v) = k$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of a vertex in G, respectively. For a vertex $v \in V(G)$ and an integer $i \geq 1$, let $d_i(v)$ denote the number of i-vertices adjacent to v. An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = \bigcup_{i=1}^{m} E(G_i)$ with $E(G_i) \cap E(G_j) = \emptyset$ for all $i \neq j$.

An *edge k-coloring* of a graph G is a function $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. The *chromatic index*, denoted by $\chi'(G)$, of a graph G is the smallest integer k such that G has an edge k-coloring. Given an edge k-coloring ϕ of G, we use $C_\phi(v)$ to denote the set of colors assigned to those edges incident to a vertex v. The coloring ϕ is called a *neighbor-distinguishing edge coloring* (an NDE-coloring for short) if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v. The *neighbor-distinguishing index* $\chi'_a(G)$ of a graph G is the smallest integer k such that G has a k-NDE-coloring. A graph G is *normal* if it contains no isolated edges. Clearly, G has an NDE-coloring if and only if G is normal. Thus, we always assume that G is normal in the following discussion.

By definition, it is easy to see that $\chi'_a(G) \geq \chi'(G) \geq \Delta(G)$ for any graph G. On the other hand, Zhang, Liu and Wang [13] proposed the following challenging conjecture, and confirmed its truth for paths, cycles, trees, complete graphs and complete bipartite graphs.

Conjecture 1. *Every connected graph G with $|V(G)| \geq 6$ has $\chi'_a(G) \leq \Delta(G) + 2$.*

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Balister et al. [2] affirmed Conjecture 1 for bipartite graphs and all graphs with $\Delta(G) = 3$. They also proved that $\chi'_a(G) \le \Delta(G) + O(\log \chi(G))$, where $\chi(G)$ is the vertex chromatic number of the graph G. This result and Brooks' Theorem imply immediately that $\chi'_a(G) \le 2\Delta(G)$ if $\Delta(G)$ is sufficiently large. Using probabilistic method, Hatami [4] showed that every graph G with $\Delta(G) > 10^{20}$ has $\chi'_a(G) \le \Delta(G) + 300$. Akbari, Bidkhori and Nosrati [1] proved that every graph G satisfies $\chi'_a(G) \le 3\Delta(G)$. Zhang, Wang and Lih [14] improved this bound to that $\chi'_a(G) \le 2.5\Delta(G) + 5$ for any graph G. For planar graphs G, Horňák, Huang and Wang [6] showed that $\chi'_a(G) \le \Delta(G) + 2$ if $\Delta(G) \ge 12$. More recently, Wang and Huang [9] further verified that if G is a planar graph with $\Delta(G) \ge 16$, then $\chi'_a(G) \le \Delta(G)+1$, and moreover $\chi'_a(G) = \Delta(G)+1$ if and only if G contains two adjacent vertices of maximum degree. This result is an extension to the result in [3], which says that if G is a planar bipartite graph with $\Delta(G) \ge 12$, then $\chi'_a(G) \le \Delta(G) + 1$. The reader is referred to [5,10–12] for other results on this direction.

In this paper, we investigate the neighbor-distinguishing index of some special graphs such as graphs with maximum degree 4 or 5 and semi-regular graphs. These results are applied to improve the upper bound of the neighbor-distinguishing index on general graphs. Here a graph G is called *semi-regular* if each edge of G is incident to at least one vertex of maximum degree. Clearly, a regular graph is a semi-regular graph, and not vice versa.

2. Graphs with $\Delta = 4$

This section is devoted to the study of the neighbor-distinguishing index of graphs with maximum degree 4.

Lemma 2.1 ([7]). *If G is a $2k$-regular graph with $k \ge 1$, then G is 2-factorizable.*

It is well-known that, given a graph G, there exists a $\Delta(G)$-regular graph H such that $G \subseteq H$. This fact, together with Lemma 2.1, implies that every graph G with $\Delta(G) = 4$ can be edge-partitioned into two subgraphs G_1 and G_2 such that $\Delta(G_i) \le 2$ for $i = 1, 2$.

In order to prove the main result in this section, i.e., Theorem 2.5, we need the following three useful consequences:

Theorem 2.2 ([14]). *If a normal graph G has an edge-partition into two normal subgraphs G_1 and G_2, then $\chi'_a(G) \le \chi'_a(G_1) + \chi'_a(G_2)$.*

Theorem 2.3 ([13]). *If P is a path of length at least two, then $\chi'_a(P) \le 3$.*

Theorem 2.4 ([2]). *If G is a graph with $\Delta(G) \le 3$, then $\chi'_a(G) \le 5$.*

Suppose that ϕ is a partial NDE-coloring of a graph G using a color set C. We call two adjacent vertices u and v *conflict under* ϕ (or simply *conflict*) if $C_\phi(u) = C_\phi(v)$. An edge uv is said to be *legally* colored if its color is different from that of its neighbors and no pair of conflict vertices is produced.

Theorem 2.5. *If G is a graph with $\Delta(G) \le 4$, then $\chi'_a(G) \le 8$.*

Proof. We prove the theorem by induction on the edge number $|E(G)|$. If $|E(G)| \le 8$, the theorem holds trivially. Let G be a graph with $\Delta(G) \le 4$ and $|E(G)| \ge 9$. If $\Delta(G) \le 3$, then the result follows from Theorem 2.4. So suppose that $\Delta(G) = 4$. The proof is split into the following cases, depending on the size of $\delta(G)$.

Case 1 $\delta(G) = 1$.

Let x be a 1-vertex adjacent to a vertex y. Let $H = G - xy$. Then H is a normal graph with $\Delta(H) \le 4$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has an 8-NDE-coloring ϕ using the color set $C = \{1, 2, \dots, 8\}$. Note that $|C_\phi(y)| = d_H(y) = d_G(y) - 1 \le 3$ and y has at most $d_G(y) - 1 \le 3$ possible conflict vertices. Thus, xy has at most $|C_\phi(y)| + 3 \le 6$ forbidden colors when colored, we can color xy with a color in $C \setminus C_\phi(y)$ such that y does not conflict with its neighbors. So an 8-NDE-coloring of G is constructed.

Case 2 $\delta(G) = 2$.

Let x be a 2-vertex with neighbors y and z. Without loss of generality, assume that $2 \le d_G(y) \le d_G(z) \le 4$. There are two possibilities to be handled.

Case 2.1 $d_G(y) = 2$.

Let w denote the neighbor of y other than x. Without loss of generality, we assume that $d_G(w) \ge 3$, for otherwise we may further consider the neighbor of w other than y until a desired vertex is found. Let $H = G - wy$. Then H is a normal graph with $\Delta(H) \le 4$ and $|E(H)| < |E(G)|$. By the induction hypothesis, H has an 8-NDE-coloring ϕ with the color set $C = \{1, 2, \dots, 8\}$. We first remove the color of xy. Since w has at most three conflict vertices and y has at most one conflict vertex, we can color yw with a color $a \in C \setminus (C_\phi(w) \cup \{\phi(xz)\})$ and xy with a color in $C \setminus \{a, \phi(xz)\}$ such that neither of x, y, w conflicts with its neighbors.