Discrete Mathematics 338 (2015) 2037–2041

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On the non-negativity of the complete **cd**-index

Neil J.Y. Fan, Liao He*

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China

ARTICLE INFO

Article history:
Received 16 September 2014
Received in revised form 30 April 2015
Accepted 2 May 2015
Available online 6 June 2015

Keywords:
Complete **cd**-index
Coxeter group
Bruhat order

ABSTRACT

The complete **cd**-index of a Bruhat interval is a non-commutative polynomial in the variables **c** and **d**, which was introduced by Billera and Brenti and conjectured to have non-negative coefficients. For a **cd**-monomial M containing at most one **d**, i.e., $M = \mathbf{c}^i$ or $M = \mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geq 0)$, Karu showed that the coefficient of M is non-negative. In this paper, we show that when $M = \mathbf{d}\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geq 0)$, the coefficient of M is non-negative.

1. Introduction

Let (W, S) be a Coxeter system and $u, v \in W$ such that $u < v$ in the Bruhat order. Billera and Brenti [2] associated the interval $[u, v]$ with a non-commutative polynomial $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ in the variables **a** and **b**. They further proved that $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ can be written as a polynomial in the variables **c** and **d**, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This new polynomial $\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is called the complete **cd**-index of $[u, v]$, which is a generalization of the **cd**-index of $[u, v]$ in the sense that the **cd**-index of $[u, v]$ is the highest degree terms of the complete **cd**-index of $[u, v]$.

The **cd**-index of an Eulerian poset is well studied, see, e.g., Billera [1] and the references therein. In particular, since a Bruhat interval is Eulerian and shellable, hence Gorenstein*, the coefficients of the **cd**-index of a Bruhat interval are non-negative, see Karu [6]. As an analogue, Billera and Brenti [2] conjectured that the complete **cd**-index still has non-negative coefficients, see also Billera [1, Conjecture 2].

Let the variables **a**, **b**, **c** have degree 1, and the variable **d** have degree 2. Blanco [3] showed that the monomials of lowest degree in the complete **cd**-index have non-negative coefficients. Suppose that M is a **cd**-monomial of degree n. If M contains at most one **d**, then Karu [7] showed that the coefficient of M is non-negative. Let us describe briefly Karu's construction. Denote by $B_n(u, v)$ the set of Bruhat paths of length $n + 1$ in the Bruhat graph of $[u, v]$. For a Bruhat path $x \in B_n(u, v)$, Karu assigned a weight $s_M(x)$ to the path x, which can be -1, 0 or 1. If there exists a flip on $[u, v]$ which is compatible with the given reflection order, then the coefficient of M is equal to the sum of weights of all the Bruhat paths in $B_n(u, v)$.

Moreover, Karu conjectured that $s_M(x) \neq -1$ for all intervals and all monomials M, which he called the flip condition. If such a condition holds, then we can construct a compatible flip, and so the non-negativity conjecture of the complete **cd**-index is true. When the **cd**-monomial M contains at most one **d**, Karu proved that the flip condition holds by using a result of Dyer [4]. Hence in this case the coefficient of M in the complete **cd**-index of $[u, v]$ is non-negative.

In this paper, we show that the coefficient of the **cd**-monomial $M = \mathbf{d}\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geq 0)$ is non-negative. Based on Karu's construction, we shall show that the number of paths with weight -1 is less than or equal to the number of paths with weight 1. To this end, we divide the involving paths into four disjoint sets according to their weights and ascent–descent sequences, and then establish two injections among these four sets of paths.

* Corresponding author.
E-mail addresses: fan@scu.edu.cn (N.J.Y. Fan), scuhlj@126.com (L. He).

http://dx.doi.org/10.1016/j.disc.2015.05.005
0012-365X/

2. Preliminary

For a Coxeter system (W, S), denote by $\ell(w)$ the length of $w \in W$. The set $T = \{wsw^{-1} \mid w \in W, s \in S\}$ is called the set of reflections of W. Let $u, v \in W$, we say that $u \prec v$ if there exists $t \in T$, such that $v = ut$ and $\ell(v) > \ell(u)$, and we say that $u < v$ if there exists a sequence of elements $u_1, u_2, \ldots, u_r \in W$ such that $u \prec u_1 \prec u_2 \prec \cdots \prec u_r \prec v$. The partial order "$<$" is called the Bruhat order of (W, S). The Bruhat graph of (W, S) is a directed graph with vertex set W and there is a directed edge from u to v if $u \prec v$.

Let $u < v$ in the Bruhat order, a Bruhat path from u to v of length $n + 1$ is a sequence

$$x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} = v). \tag{1}$$

We label the edge $x_i \prec x_{i+1}$ by the reflection $t_i = x_i^{-1}x_{i+1}$ for $i = 0, 1, \ldots, n$. Let $B_n(u, v)$ denote all the Bruhat paths from u to v of length $n + 1$. Denote by $B(u, v) = \bigcup_{n\geq 0} B_n(u, v)$ the set of all Bruhat paths from u to v.

Recall that a reflection order $(\mathcal{O}, <_T)$ is a total order defined on the set of reflections T, see [5]. The reverse of the order $\mathcal{O}$, denoted by $\overline{\mathcal{O}}$, is also a reflection order. In the sequel, we will always use the reflection order $(\mathcal{O}, <_T)$. We say that the path x in (1) is increasing (resp., decreasing), if $t_0 <_T t_1 <_T \cdots <_T t_n$ (resp., $t_n <_T t_{n-1} <_T \cdots <_T t_0$). The following result is due to Dyer [4].

Theorem 2.1 ([4]). *Let $x = (u = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = v)$ be an increasing path in $B_n(u, v)$, and $y = (u = y_0 \prec y_1 \prec \cdots \prec y_m \prec y_{m+1} = v)$ be a decreasing path in $B_m(u, v)$. Then we have*

$$x_0^{-1}x_1 <_T y_0^{-1}y_1, \qquad y_m^{-1}y_{m+1} <_T x_n^{-1}x_{n+1}.$$

For the Bruhat path $x \in B_n(u, v)$ in (1), define the ascent–descent sequence of x by

$$\omega(x) = \beta_1\beta_2 \cdots \beta_n,$$

where

$$\beta_i = \begin{cases} \mathbf{a}, & \text{if } t_{i-1} <_T t_i; \\ \mathbf{b}, & \text{if } t_i <_T t_{i-1}. \end{cases}$$

In [2], Billera and Brenti associate each interval $[u, v]$ with a non-homogeneous polynomial $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$ by summing over the ascent–descent sequences of all the Bruhat paths in $B(u, v)$. That is,

$$\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}) = \sum_{x\in B(u,v)} \omega(x).$$

It can be shown that $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ is independent of the given reflection order. Moreover, $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ can be rewritten as a polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$, $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. That is,

$$\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \tilde{\psi}_{u,v}(\mathbf{a} + \mathbf{b}, \mathbf{ab} + \mathbf{ba}) = \tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}).$$

This new polynomial $\tilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is called the complete **cd**-index of $[u, v]$.

Now we proceed to recall some definitions and results in [7].

For an **ab**-monomial $M(\mathbf{a}, \mathbf{b})$, denote by $\overline{M} = M(\mathbf{b}, \mathbf{a})$ the **ab**-monomial obtained by exchanging **a** and **b** in M. This operator is an involution in the non-commutative ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$.

Definition 2.2. A flip $F = F_{u,v}$ on $[u, v]$ is an involution

$$F_{u,v} : B(u, v) \to B(u, v),$$

such that $\omega(F(x)) = \overline{\omega(x)}$ for all $x \in B(u, v)$.

Note that since $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ is independent of the reflection order, when we use the orders $\mathcal{O}$ and $\overline{\mathcal{O}}$ to compute $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ respectively, we obtain $\tilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}) = \tilde{\phi}_{u,v}(\mathbf{b}, \mathbf{a})$. That is to say, any flip $F_{u,v}$ on $[u, v]$ has no fixed points.

We fix a flip for every interval in advance. Let $1 \leq m \leq n$ and

$$x = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_m \prec x_{m+1} \prec \cdots \prec x_n \prec x_{n+1} = v).$$

After applying the flip $F_{x_m,v}$ to x, we get

$$y = (u = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_m \prec y_{m+1} \prec \cdots \prec y_n \prec y_{n+1} = v).$$

If $\omega(x) = \beta_1 \cdots \beta_m \cdots \beta_n$, then $\omega(y) = \beta_1 \cdots \beta_{m-1}\alpha_m\overline{\beta}_{m+1} \cdots \overline{\beta}_n$, where α_m can be either **a** or **b**. Define

$$s_{m,\mathbf{a}}(x) = \begin{cases} 1, & \text{if } \beta_m = \mathbf{a}; \\ 0, & \text{otherwise.} \end{cases}$$