Discrete Mathematics 338 (2015) 2042–2050

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Hamiltonian claw-free graphs with locally disconnected vertices

Runli Tian[a,b], Liming Xiong[a,*]

[a] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China

[b] School of Science, Central South University of Forestry and Technology, Changsha 410004, PR China

ARTICLE INFO

Article history:
Received 23 June 2013
Received in revised form 3 April 2015
Accepted 20 April 2015
Available online 6 June 2015

Keywords:
Claw-free graph
Hamiltonian
Closure
Locally disconnected vertex
Singular edge

ABSTRACT

An edge of G is *singular* if it does not lie on any triangle of G; otherwise, it is *non-singular*. A vertex u of a graph G is called *locally connected* if the induced subgraph $G[N(u)]$ by its neighborhood is connected; otherwise, it is called *locally disconnected*.

In this paper, we prove that if a connected claw-free graph G of order at least three satisfies the following two conditions: For each locally disconnected vertex v of G with degree at least 3, there is a nonnegative integer s such that v lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-3$ locally connected vertices; for each locally disconnected vertex v of G with degree 2, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s-2$ locally connected vertices and such that the subgraph induced by those vertices of C that have degree two in G is a path or a cycle, then G is Hamiltonian, and it is best possible in some sense.

Our result is a common extension of two known results in Bielak (2000) and in Li (2002) ; hence also of the results in Oberly and Sumner (1979) and in Ryjáček (1990).

1. Introduction

We consider only finite undirected simple graphs, unless otherwise stated. For terminology and notation not defined in this paper we refer to [9].

If H is a graph, then the *line graph* of H, denoted by $L(H)$, is the graph with $E(H)$ as its vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. For a family $\mathcal{F}$ of a connected graphs, a graph is called $\mathcal{F}$-free if it contains no induced copies of any member of $\mathcal{F}$. The graph $K_{1,3}$ is called a *claw*. It is a well-known fact that every line graph is claw-free, hence the class of the claw-free graphs can be considered as a natural generalization of the class of line graphs.

The neighborhood of a vertex v in G is denoted by $N_G(v)$. Denote $N_G[v] = N_G(v) \cup \{v\}$. A vertex v of G is *locally connected* if $G[N_G(v)]$ is connected; otherwise, it is *locally disconnected*. Let $LC(G)$ denote the set of all locally connected vertices of G. A graph G is called *locally connected* if every vertex of G is locally connected, *i.e.*, $LC(G) = V(G)$. Oberly and Sumner proved the following well-known result.

Theorem 1 (*Oberly and Sumner* [5]). *Every connected, locally connected claw-free graph on at least three vertices is Hamiltonian.*

* Corresponding author.
E-mail address: lmxiong@bit.edu.cn (L. Xiong).

http://dx.doi.org/10.1016/j.disc.2015.04.020

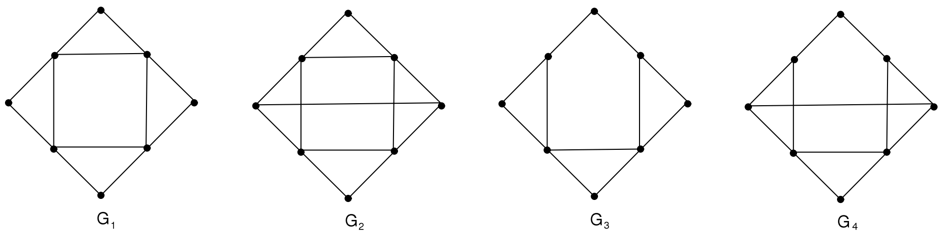

Fig. 1. The graphs G_1, G_2, G_3 and G_4.

We say that a vertex v of a graph G is N_2-*locally connected* if the subgraph of G induced by the edge set $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$ is connected. A graph G is called N_2-*locally connected* if every vertex of G is N_2-locally connected. It follows from the definitions that every locally connected graph is N_2-locally connected, but the converse is not true.

In 1990, Ryjáček [7] considered the graphs with some locally disconnected vertices in claw-free graphs and strengthened Theorem 1 by using this concept of N_2-locally connected. He showed that every connected N_2-locally connected claw-free graph G with $\delta(G) \geq 2$ satisfying that G has no induced subgraph H isomorphic to either G_1 or G_2 (in Fig. 1) such that every vertex of degree 4 in H is locally disconnected in G is Hamiltonian. Bielak later improved this result by weakening the condition. Their result can be restated as the following theorem, where $V_i(G) = \{x : d_G(x) = i\}$ and $V_{\geq i}(G) = \{x : d_G(x) \geq i\}$.

Theorem 2 (*Bielak [1]*)**.** *Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geq 2$ such that*

(1) *every induced subgraph H of G isomorphic to either G_1 or G_2 (in Fig. 1) has at least one locally connected vertex of G in $V_3(H) \cup V_4(H)$.*

Then G is Hamiltonian.

In this paper, we shall continue to extend the above result which will need some notation. We say that a vertex v of a graph G is N^2-*locally connected* if the subgraph of G induced by the vertices $\{x \in V(G) : 1 \leq d(x, v) \leq 2\}$ is connected, where $d(x, v)$ denotes the distance between x and v. A graph G is called N^2-*locally connected* if every vertex of G is N^2-locally connected. Obviously, every N_2-locally connected graph is N^2-locally connected, but the converse is not generally true.

Theorem 3. *Let G be a connected, N^2-locally connected claw-free graph with $\delta(G) \geq 2$ satisfying*

(2) *every induced subgraph isomorphic to one of $\{G_1, G_2, G_3, G_4\}$ (in Fig. 1) has at least one locally connected vertex of G in $V_3(H) \cup V_4(H)$.*

Then G is Hamiltonian.

From Theorem 3, one can obtain the following known result immediately.

Corollary 4 (*Li [4]*)**.** *Every connected N^2-locally connected $\{G_1, G_2, G_3, G_4, K_{1,3}\}$-free graph G with $\delta(G) \geq 2$ is Hamiltonian.*

Let G_0 be the graph obtained from some graph G_i in Fig. 1 by joining all vertices of an additional complete graph of arbitrarily larger order to some vertex of degree four or three in G_i and to its neighbors. Then G_0 satisfies the conditions of Theorem 3 but not Corollary 4. This shows that Theorem 3 is stronger than Corollary 4.

Motivated by the above observation, in this paper, we intend to generality them by avoiding using the concept of N_2-(or N^2-)connected and use certain technical conditions on locally disconnected vertices instead. Here we need divide all edges of the graphs into two kinds of edges: An edge e of G is *singular* if it does not lie on any triangle of G; otherwise, it is *non-singular*. We have the following result that can deduce Theorem 3, as showed in Section 4.

Theorem 5. *Let G be a connected claw-free graph of order at least three such that*

(i) *for each locally disconnected vertex v of degree at least 3 in G, there is a nonnegative integer s such that v lies on an induced cycle of length at least four with at most s non-singular edges and with at least $s - 3$ locally connected vertices;*
(ii) *for each locally disconnected vertex v of degree 2 in G, there is a nonnegative integer s such that v lies on an induced cycle C with at most s non-singular edges and with at least $s - 2$ locally connected vertices and such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.*

Then G is Hamiltonian.

In Section 2, we shall present Ryjáček's closure concept in claw-free graphs and some auxiliary results, which are then applied to the proof of our main result in Section 3. Section 4 is devoted to the proof of Theorem 3. In the last section, we discuss the sharpness of our main results, point out a flaw in the original proof of Corollary 4 and show that Theorem 5 is stronger than Theorem 3, and hence also than Corollary 4.