Discrete Mathematics 338 (2015) 2072–2079

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The b-chromatic index of graphs[☆]

Victor A. Campos [a], Carlos V. Lima [b,a], Nicolas A. Martins [a], Leonardo Sampaio [c,a], Marcio C. Santos [d,a], Ana Silva [a,*]

[a] *ParGO Research Group, Universidade Federal do Ceará, Av. Mister Hull s/n - Pici, Fortaleza/CE, Brazil*
[b] *COPPE, Universidade Federal do Rio de Janeiro, Av. Horácio Macedo, 2030, Bloco G - Cidade Universitária, Rio de Janeiro/RJ, Brazil*
[c] *Universidade Estadual do Ceará, Av. Dr. Silas Muguba, 1700 - Itaperi, Fortaleza/CE, Brazil*
[d] *Université de Techologie de Compiégne (UTC), Rue du Docteur Schweitzer, 60200, Compiègne, France*

ARTICLE INFO

Article history:
Received 16 September 2014
Received in revised form 23 April 2015
Accepted 28 April 2015
Available online 6 June 2015

Keywords:
b-chromatic index
Edge coloring
Totally Unimodular matrices
Caterpillars
Trees

ABSTRACT

A b-coloring of the vertices of a graph is a proper coloring where each color class contains a vertex which is adjacent to at least one vertex in each other color class. The b-chromatic number of G is the maximum integer $b(G)$ for which G has a b-coloring with $b(G)$ colors. This problem was introduced by Irving and Manlove (1999), where they showed that computing $b(G)$ is $\mathcal{NP}$-hard in general and polynomial-time solvable for trees. A natural question that arises is whether the edge version of this problem is also $\mathcal{NP}$-hard or not. Here, we prove that computing the b-chromatic index of a graph G is $\mathcal{NP}$-hard, even if G is either a comparability graph or a C_k-free graph, and give partial results on the complexity of the problem restricted to trees, more specifically, we solve the problem for caterpillars graphs. Although solving problems on caterpillar graphs is usually quite simple, this problem revealed itself to be unusually hard. The presented algorithm uses a dynamic programming approach that combines partial solutions which are proved to exist if, and only if, a particular polyhedron is non-empty.

1. Introduction

Let G be a simple graph[1] and suppose that we have a proper coloring of G for which there exists a color class C such that every vertex v in C is not adjacent to at least one other color class; then we can separately change the color of each vertex in C to obtain a proper coloring with fewer colors. This heuristic, called here b-heuristic, can be applied iteratively, but we cannot expect to reach the chromatic number of G, since the coloring problem is $\mathcal{NP}$-hard.

On the basis of this idea, Irving and Manlove introduced the notion of b-coloring in [12]. Intuitively, a b-coloring is a proper coloring that cannot be improved by the b-heuristic, and the b-chromatic number $b(G)$ measures the worst possible such coloring. Finding $b(G)$ was proved to be $\mathcal{NP}$-hard in general graphs [12], and remains so even when restricted to bipartite graphs [16] or to chordal graphs [10]. However, this problem is polynomial when restricted to some graph classes, including trees [12], cographs and P_4-sparse graphs [3], P_4-tidy graphs [20], cacti [18], some power graphs [5–7], Kneser graphs [9,14],

☆ Partially supported by CAPES, FUNCAP and CNPq/Brazil.
* Corresponding author.
E-mail addresses: campos@lia.ufc.br (V.A. Campos), gclima@cos.ufrj.br (C.V. Lima), nicolas@lia.ufc.br (N.A. Martins), leonardo.sampaio@uece.br (L. Sampaio), marciocs5@lia.ufc.br (M.C. Santos), anasilva@mat.ufc.br (A. Silva).
[1] The graph terminology used in this paper follows [2].

http://dx.doi.org/10.1016/j.disc.2015.04.026

some graphs with large girth [8,15,17], etc. Also, some other aspects of the problem were studied, as for example, the b-spectrum of a graph [1], and b-perfect graphs [11].

In this article, we propose to study a natural variation of the b-coloring problem, coloring the edges of a graph under the same constraints. In fact, we will investigate the b-coloring of the line graph of some classes of graphs. More formally, a *proper coloring (with k colors)* of a graph is a function $\psi : V(G) \to \{1, \ldots, k\}$ such that no two adjacent vertices have the same color (function value). For $X \subseteq V(G)$, denote by $\psi(X)$ the set $\{\psi(v) \mid v \in X\}$. We say that a vertex v *realizes color* $\psi(v)$ if $\psi(N(v))$ contains every color distinct from $\psi(v)$. We also call v a *b-vertex* and say that $\psi(v)$ is *realized (on v)*. A *b-coloring of G* is a proper coloring ψ such that each color is realized. The *b-chromatic number of G* is the maximum integer $b(G)$ for which G has a b-coloring with $b(G)$ colors.

In [12], Irving and Manlove also introduced a simple upper bound for $b(G)$, defined as follows. The *m-degree of G* is the maximum integer k for which there are at least k edges of degree at least $k-1$; we denote it by $m(G)$. It is easy to see that

$$\chi(G) \le b(G) \le m(G).$$

We mention that, up to our knowledge, only one other work has been done on this metric. In [13], Jakovac and Peterin investigate graphs whose line graphs have b-chromatic number inferior to their m-degree, and prove that graphs whose line graphs are cubic have b-chromatic number equal to 5, with the exception of four line graphs: K_4, $K_{3,3}$, the prism over K_3, and the cube Q_3. They also claim to have proved that the b-chromatic number of the line graph G of a tree is either $m(G)$, or $m(G)-1$. However, in [17] the authors show that $m(G)-b(G)$ can be arbitrarily large when G is the line graph of a tree. Nevertheless, in Section 3, we prove that this difference is at most 1 when the tree is a caterpillar. Despite being a simple class of graphs, the algorithm found is not quite as simple, and combines a Linear Programming model whose coefficient matrix is proved to be Totally Unimodular (TU), and a dynamic programming algorithm. For general trees, some partial results are presented in [19], and the decision problem for fixed k is proved to be polynomial-time solvable [10,19]. We mention that b-coloring the line graph of a tree is equivalent to b-coloring a claw-free block graph, which are contained in the class of chordal graphs. Computing $b(G)$ when G is chordal is $\mathcal{NP}$-hard and, up to now, nothing was known about the b-chromatic number of subclasses of chordal graphs. Finally, in Section 2, we prove that deciding if $b(G)$ equals $m(G)$ is $\mathcal{NP}$-complete, even if G is the line graph of either a comparability graph or a C_k-free graph.

In the remaining text, we use the following notation and terminology. Consider a simple graph G, and let $u \in V(G)$. The *neighborhood of u in G* is the set of vertices adjacent to u, and is denoted by $N_G(u)$. The *closed neighborhood of u in G* is the set $N_G(u) \cup \{u\}$, and is denoted by $N_G[u]$. The *degree* of $u \in V(G)$ is the cardinality of $N(u)$, and is denoted by $d_G(u)$; analogously, the *degree of an edge* $e \in E(G)$ is the number of edges adjacent to e, and is denoted by $d_G(e)$. In every case, the subscript can be omitted if there is no ambiguity. Now, consider $X \subseteq V(G)$. Then, $N(X)$ denotes the subset $(\bigcup_{x \in X} N(x)) \setminus X$, while $N[X]$ denotes the subset $N(X) \cup X$. Also, given a proper coloring ψ of $V(G)$, we denote by $\psi(X)$ the set $\{\psi(x) \mid x \in X\}$. Finally, the subset of all vertices of G with degree at least $m(G)-1$ is denoted by $D(G)$ (these are the "candidates" for b-vertices).

2. $\mathcal{NP}$-completeness

In this section, for clarity reasons, we present the problem as an edge b-coloring. Consider the adaptation of the problem to edge-coloring. The corresponding values of $b(G)$ and $m(G)$ are denoted by $b'(G)$ and $m'(G)$, respectively. We consider the following problems:

Edge Coloring
Instance: A graph G and an integer k, $k \ge 3$.
Question: Is there a proper edge coloring of G with k colors?

Edge B-Coloring
Instance: A graph G.
Question: Is $b'(G)$ equal to $m'(G)$?

A graph is called k-regular if each of its vertices has degree k; a comparability graph is a graph whose edges can be transitively oriented; and a graph is called C_t-free if it has no induced cycle of length t. The Problem **Edge Coloring** is $\mathcal{NP}$-complete even when G is a k-regular graph and is either a comparability graph or a C_t-free graphs[4]. We prove that this problem can be reduced to the Problem **Edge B-Coloring**.

Theorem 1. *Edge B-Coloring is $\mathcal{NP}$-complete, even if G is either a comparability graph or a C_k-free graph, for $k \ge 4$.*

Proof. Denote by V the vertex set of G and by n the cardinality of V. To verify if an edge coloring is an edge b-coloring with $m'(G)$ colors can be done in polynomial time and so the problem is in $\mathcal{NP}$. We show a reduction from **Edge Coloring** of d-regular graphs in order to prove $\mathcal{NP}$-completeness. Also, we show that the construction is closed under the subclasses of comparability and of C_k-free graphs, for $k \ge 4$. Since **Edge Coloring** is $\mathcal{NP}$-complete even when restricted to instances of type (G, d), where G is a d-regular graph which is also either a comparability graph or a C_k-free graph [4], the theorem follows.

Consider a d-regular graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Let H be the graph constructed from G as follows. Add vertices w, w' and edges $ww', wv_1, \ldots, wv_n$ to H. Finally, for each $i \in \{1, \ldots, d\}$, add vertices $w_i, x_1^i, \ldots, x_n^i$ and edges $w'w_i$ and $w_ix_j^i$, for all $j \in \{1, \ldots, n\}$. Fig. 1 shows graph H.