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Orthogonal matchings revisited

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ABSTRACT

Let G be a graph on n vertices, which is an edge-disjoint union of ms-factors, that is, s regular spanning subgraphs. Alspach first posed the problem that if there exists a matching M of m edges with exactly one edge from each 2-factor. Such a matching is called orthogonal because of applications in design theory. For $s = 2$, so far the best known result is due to Stong in 2002, which states that if $n \geq 3m-2$, then there is an orthogonal matching. Anstee and Caccetta also asked if there is a matching M of m edges with exactly one edge from each s-factor? They answered yes for $s \geq 3$. In this paper, we get a better bound and prove that if $s = 2$ and $n \geq 2\sqrt{2}m+4.5$ (note that $2\sqrt{2} \leq 2.825$), then there is an orthogonal matching. We also prove that if $s = 1$ and $n \geq 3.2m - 1$, then there is an orthogonal matching, which improves the previous bound ($3.79m$).

1. Introduction and notation

We use [4] for terminology and notations not defined here and consider simple undirected graphs only. Let $G = (V, E)$ be a graph. For a subgraph H of G, let $|H|$ denote the order of H, i.e. the number of vertices of H and let $\|H\|$ denote the size of H, that is, the number of edges of H. If a vertex u is an end vertex of an edge e, we write $u \in e$.

Let G be a graph on n vertices, which is an edge-disjoint union of ms-factors, that is, s regular spanning subgraphs. In 1988, Alspach [1] first posed the problem that if there exists a matching M of m edges with exactly one edge from each 2-factor. Such a matching is called *orthogonal* because of applications in design theory. A matching M is *suborthogonal* if there is at most one edge from each s-factor. Alspach, Heinrich and Liu [2] proved that the answer is affirmative if $n \geq 4m-5$. Kouider and Sotteau improved this bound to $3.23m$. In 2002, Stong [17] further improved this bound and proved the following result.

Theorem 1.1 ([17]). *Let G be a $2m$-regular graph with $n \geq 3m - 2$. Then for any decomposition of $E(G)$ into m 2-factors $F_1, F_2, \ldots, F_m$, there is an orthogonal matching.*

The problem with $s = 2$ and all the 2-factors being hamiltonian cycles was raised by Caccetta and Mardiyono [5] and Chung (referred to in [12]) but apparently the extra condition is no help.

In 1998, Anstee and Caccetta [3] asked if there is a matching M of m edges with exactly one edge from each s-factor in the cases of $s = 1$ and $s \geq 3$? For $s \geq 3$, the answer is yes (see [3]).

For $s = 1$, the answer is negative: let G be a complete graph K_{m+1} (m is even) which is an edge disjoint union of m 1-factors, however, the size of maximum matching is at most $\frac{m}{2}$. Indeed, it is best possible, see [11]. But how about when we restrict ourselves to large graph? Wang, Liu and Liu [20] proved the following result.

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Theorem 1.2 ([20]). *Let G be an m-regular graph with $n \geq 3.79m$. Then for any decomposition of $E(G)$ into m 1-factors $F_1, F_2, \ldots, F_m$, there is an orthogonal matching.*

In particular, if G is $K_{m,m}$ and is a union of m 1-factors $F_1, F_2, \ldots, F_m$, then G corresponds to a Latin square, where entry a_{ij} is l if edge $(u_i, v_j) \in F_l$. Now our desired matching corresponds to a transversal. Hatami and Shor [9] proved that if $K_{m,m}$ is a union of m 1-factors $F_1, F_2, \ldots, F_m$, then there is a matching M of p edges with at most one edge from any 1-factor with $p = m - O(\log m)^2$.

If G is assigned an arbitrary edge-coloring (not necessarily proper), then we say that G is an *edge-colored graph*. A subgraph H of an edge-colored graph G is called *rainbow* (also *heterochromatic*, *multicolored*, *polychromatic*) if its edges have distinct colors. The *minimum color degree* of G is the smallest number of distinct colors on the edges incident with a vertex over all vertices. Recently, the study of rainbow paths and cycles under minimum color degree condition has received much attention, see [6,15]. For rainbow matchings under minimum color degree condition, see [11,10,16,13,14,19].

In any decomposition of $E(G)$ into ms-factors, we can construct an edge-colored graph by giving each s-factor a color. Then a rainbow matching of G corresponds to a suborthogonal matching of G. In particular, when $s = 1$, the edge-colored graph obtained above is properly edge-colored. For rainbow matchings in properly edge-colored graphs, see [7,8,18,21].

In this paper, we improve the bounds in Theorems 1.1 and 1.2 and get the following results.

Theorem 1.3. *Let G be an m-regular graph with $n \geq 3.2m - 1$. Then for any decomposition of $E(G)$ into m 1-factors $F_1, F_2, \ldots, F_m$, there is an orthogonal matching.*

Theorem 1.4. *Let G be a $2m$-regular graph with $n \geq 2\sqrt{2}m + 4.5$. Then for any decomposition of $E(G)$ into m 2-factors $F_1, F_2, \ldots, F_m$, there is an orthogonal matching.*

2. Proof of main results

We prove our conclusions by contradiction. Firstly, when $m = 1$ and $m = 2$, the proof is trivial. If Theorems 1.3 and 1.4 are false, then there exists a minimal m, such that there is no a rainbow matching of size m for G. We construct an edge-colored graph by giving each 1-factor (in Theorem 1.3), 2-factor (in Theorem 1.4) a color from $\{1, 2, \ldots, m\}$. For an edge $e \in E(G)$, let $c(e)$ denote the color of e. For a subgraph H of G, let $C(H) = \{c(e) \mid e \in E(H)\}$. By the minimality of m, G has a rainbow matching of size $m - 1$. For simplicity, let $p = m - 1$ and $n = |G|$. We define a *good configuration* $H_p = M_1 \cup M_2 \cup M_3 \cup F$ as follows (see Fig. 1). Note that the blue edges in the figure are colored m.

(a) For some integer $k \geq 0$, $M_1 = \{e_i\ (e_i = u_i v_i) : i = 1, 2, \ldots, k\}$ and $M_2 = \{f_i : i = 1, 2, \ldots, k\}$ are two vertex-disjoint rainbow matchings of G with $c(e_i) = c(f_i)$.

(b) $M_3 = \{g_i\ (g_i = u_i v_i) : i = k + 1, \ldots, p\}$ is a rainbow matching, which is vertex-disjoint from $M_1 \cup M_2$ and $c(g_i) \neq c(e_j)$ for $1 \leq j \leq k < i \leq p$.

For abbreviation, let G_1 denote the subgraph induced by $V(G) \backslash V(M_1 \cup M_2 \cup M_3)$. Without loss of generality, we assume that $C(M_1 \cup M_3) = \{1, 2, \ldots, m - 1\}$.

(c) $F = \{h_i\ (h_i = v_i z_i) : i = k + 1, \ldots, k + t\}$ is a matching, vertex-disjoint from $M_1 \cup M_2$, $h_i \cap M_3 = \{v_i\} \in g_i$, and $c(h_i) = m$.

We choose a good configuration $H_p = M_1 \cup M_2 \cup M_3 \cup F$ satisfying the following conditions:

(1) $k = \|M_1\|$ is maximum;

(2) subject to (1), F is maximal, that is, F covers the maximum number of vertices of M_3.

Claim 2.1. *If $u \in V(G_1)$ and $c(uv) = m$, then $v \in V(M_3)$.*

Proof. By symmetry, we may assume that $v \notin V(M_2)$. If $v \notin V(M_3)$, then $M_2 \cup M_3 \cup uv$ is an orthogonal matching of G, which is a contradiction. □

Claim 2.2. *If $u \in V(e_i \cup f_i)$ and $c(uv) = m$, where $v \notin V(M_3)$, then $v \in V(e_i \cup f_i)$.*

Proof. Suppose to the contrary that $v \notin V(e_i \cup f_i)$. By symmetry and without loss of generality, we may assume that $u, v \notin V(M_2)$. Since $c(uv) = m$, $M_2 \cup M_3 \cup uv$ is an orthogonal matching, which is a contradiction. □

If there is an edge uv such that $u, v \in V(e_i \cup f_i)$ and $c(uv) = m$, then we call $e_i \cup f_i$ a *nice pair*. Let q denote the number of nice pairs in $M_1 \cup M_2$. Without loss of generality, we assume that the nice pairs are $\{e_1 \cup f_1, \ldots, e_q \cup f_q\}$ and we call $c(e_i)$ a *nice color*, for $i = 1, 2, \ldots, q$. Let n_1 be the number of edges uv such that $u \in V(M_3)$, $v \in V(G) \backslash V(M_3)$ and $c(uv) = m$. Note that each vertex is incident with at least one edge with color m since each color induces a 1-factor (in Theorem 1.3) or 2-factor (in Theorem 1.4).

Claim 2.3. *We have that $V(H_p) = V(G)$.*