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Note

On a link between Dirichlet kernels and central multinomial coefficients

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ABSTRACT

The central coefficients of powers of certain polynomials with arbitrary degree in x form an important family of integer sequences. Although various recursive equations addressing these coefficients do exist, no explicit analytic representation has yet been proposed. In this article, we present an explicit form of the integer sequences of central multinomial coefficients of polynomials of even degree in terms of finite sums over Dirichlet kernels, hence linking these sequences to discrete nth-degree Fourier series expansions. The approach utilizes the diagonalization of circulant Boolean matrices, and is generalizable to all multinomial coefficients of certain polynomials with even degree, thus forming the base for a new family of combinatorial identities.

1. Introduction

Let $k \in \mathbb{N}$, $k \geq 1$ and

$$P_{2k}(x) = 1 + x + x^2 + \cdots + x^{2k} \tag{1}$$

be a finite polynomial of even degree in x. Using the multinomial theorem and collecting terms with the same power in x, the nth power of $P_{2k}(x)$ with $n \in \mathbb{N}$, $n \geq 1$ is then given by

$$P_{2k}(x)^n = \left(1 + x + x^2 + \cdots + x^{2k}\right)^n = \sum_{l=0}^{2kn} p_{l,2k}^{(n)} x^l, \tag{2}$$

where $p_{l,2k}^{(n)}$ denotes the multinomial coefficient (e.g., see [4, Definition B, p. 28]) given by

$$p_{l,2k}^{(n)} = \sum_{n_i} \binom{n}{n_0, n_1, \ldots, n_{2k}} \tag{3}$$

$\forall l \in [0, 2kn]$, where in the last equation the sum runs over $n_i \in [0, n]\ \forall i \in [0, 2k]$ with $n_0 + n_1 + \cdots + n_{2k} = n$ and $n_1 + 2n_2 + \cdots + 2kn_{2k} = l$. The central $(2k+1)$-nomial coefficients $M^{(2k,n)}$, e.g., the central trinomial ($k = 1$), pentanomial

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($k = 2$) or heptanomial ($k = 3$) coefficients, are then given by

$$M^{(2k,n)} = p^{(n)}_{\lfloor 2kn/2 \rfloor,2k} \equiv p^{(n)}_{kn,2k}. \tag{4}$$

Eqs. (3) and (4) provide a definition of the central coefficients in terms of sums over products of binomial coefficients. However, their explicit calculation constitutes a numerically non-trivial problem, especially for large k and n, as it involves sums over specific partitions of integer numbers. Although various recursive equations addressing these coefficients do exist, and the Almkvist–Zeilberger algorithm [2] allows for a systematic derivation of recursions for multinomial coefficients in the general case, no explicit analytical representation of these coefficients has yet been proposed in the literature.

Mathematically, the central multinomial coefficients $M^{(2k,n)}$ are linked to the number of closed walks of length n in random graphs, and recently an approach has been proposed which translates this combinatorially hard problem into one of taking powers of a specific type of circulant Boolean matrices [8, Equations (7) and (8)]. In this paper, we detail this approach, and prove a simple relation between central $(2k + 1)$-nomial coefficients defined in (4) and finite sums over Dirichlet kernels of fractional angles (for Dirichlet kernels, e.g., see [6], and Chapter I, §29 in [3]). This explicit analytical representation not only allows for a fast numerical evaluation of central multinomial coefficients, but also for an explicit construction of the whole class of sequences of central multinomial coefficients (see the On-Line Encyclopedia of Integer Sequences, OEIS, [10]; e.g., OEIS A002426, A005191, A025012, A025014).

2. A trace formula

Let $N = 2kn + 1$ with $n, k \in \mathbb{N}$. Consider the $N \times N$ circulant matrix

$$\mathbf{A}_{N,2k} = \text{circ}\Big\{(1, \overbrace{1, \ldots, 1}^{2k}, 0, \ldots, 0)\Big\} = \text{circ}\left\{\left(\sum_{l=0}^{2k} \delta_{j,1+l \bmod N}\right)_j\right\}. \tag{5}$$

Multiplying $\mathbf{A}_{N,2k}$ by a vector $\mathbf{x} = (1, x, x^2, \ldots, x^{2kn})$ will yield the original polynomial as the first element in the resulting vector $\mathbf{A}_{N,2k}\mathbf{x}$. Similarly, taking the n'th ($n' \le n$) power of $\mathbf{A}_{N,2k}$ and multiplying the result with $\mathbf{x}$ will yield $P_{2k}(x)^{n'}$ as first element. Thus $\mathbf{A}^{n'}_{N,2k}$ will contain the sequence of multinomial coefficients $p^{(n')}_{l,2k}$ in its first row. Moreover, as the power of a circulant matrix is again circulant, this continuous sequence of non-zero entries in a given row will shift by one column to the right on each subsequent row, and wraps around once the row-dimension N is reached. This behavior will not change even if one introduces a shift by m columns of the sequence of 1's in $\mathbf{A}_{N,2k}$, as this will correspond to simply multiplying the original polynomial by x^m. Such a shift, however, will allow, when correctly chosen, to bring the desired central multinomial coefficients on the diagonal of $\mathbf{A}^{n'}_{N,2k}$.

We can formalize this approach in the following.

Lemma 1. *Let*

$$\mathbf{A}^{(m)}_{N,2k} = \text{circ}\left\{\left(\sum_{l=0}^{2k} \delta_{j,1+(m+l) \bmod N}\right)_j\right\} \tag{6}$$

with $m \in \mathbb{N}_0$ be circulant Boolean square matrices of dimension $N = 2kn + 1$ with $k, n \in \mathbb{N}$. The central $(2k + 1)$-nomial coefficients are given by

$$M^{(2k,n)} = \frac{1}{N} \text{Tr}\left(\mathbf{A}^{(N-k)}_{N,2k}\right)^n, \tag{7}$$

where $\text{Tr}(\mathbf{A})$ denotes the trace of matrix $\mathbf{A}$.

Proof. Let $\mathbf{B} = \text{circ}\big\{(0, 1, 0, \ldots, 0)\big\}$ be a $N \times N$ cyclic permutation matrix. Note that $\mathbf{B}$ has the following properties:

$$\begin{aligned} &\mathbf{B}^0 \equiv I = \mathbf{B}^N \\ &\mathbf{B}^m = \mathbf{B}^r \quad \text{with } r = m \mod N \\ &\mathbf{B}^n\mathbf{B}^m = \mathbf{B}^{(nm) \mod N}, \end{aligned} \tag{8}$$

where $\mathbf{I}$ denotes the identity matrix of order N. The set of powers of the cyclic permutation matrix, $\{\mathbf{B}^m\}$, $m \in [0, N]$, then acts as a basis for the circulant matrices $\mathbf{A}^{(m)}_{N,2k}$ defined in (6) (e.g., see [11, Section 1.10] and [5] for a thorough introduction into circulant matrix algebra).

Let us first consider the case $m = 0$. It can easily be shown that

$$\mathbf{A}^{(0)}_{N,2k} = \mathbf{I} + \sum_{l=1}^{2k} \mathbf{B}^l.$$