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Linearized Wenger graphs

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ABSTRACT

Motivated by recent extensive studies on Wenger graphs, we introduce a new infinite class of bipartite graphs of a similar type, called linearized Wenger graphs. The spectrum, diameter and girth of these linearized Wenger graphs are determined.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of order q such that p is prime and $q = p^e$ a prime power. All graph theory notions can be found in Bollobás [1]. Recently, a class of bipartite graphs called *Wenger graphs* which are defined over $\mathbb{F}_q$ has attracted a lot of attention because of their nice graphical properties [2,6,7,12–16]. For example, the number of edges of these graphs meets asymptotically in magnitude the upper bounds of Turán number of the cycle with length 4, 6, 10 [16]. The original definition was introduced by Wenger [16] for p-regular bipartite graphs and then was extended by Lazebnik and Ustimenko [6] for arbitrary prime power q. An equivalent representation of these graphs appeared later in Lazebnik and Viglione [9] and then a more general class of graphs was defined in [14], on which we concentrate in this paper.

Let $m \geq 1$ be a positive integer and $g_k(x, y) \in \mathbb{F}_q[x, y]$ for $2 \leq k \leq m+1$. Let $\mathfrak{P} = \mathbb{F}_q^{m+1}$ and $\mathfrak{L} = \mathbb{F}_q^{m+1}$ be two copies of the $(m+1)$-dimensional vector space over $\mathbb{F}_q$, which are called the point set and the line set respectively. If $a \in \mathbb{F}_q^{m+1}$, then we write $(a) \in \mathfrak{P}$ and $[a] \in \mathfrak{L}$. Let $\mathfrak{G} = G_q(g_2, \ldots, g_{m+1}) = (V, E)$ be the bipartite graph with vertex set $V = \mathfrak{P} \cup \mathfrak{L}$ and the edge set E is defined as follows: there is an edge from a point $P = (p_1, p_2, \ldots, p_{m+1}) \in \mathfrak{P}$ to a line $L = [l_1, l_2, \ldots, l_{m+1}] \in \mathfrak{L}$, denoted by $P \sim L$, if the following m equalities hold:

$$l_2 + p_2 = g_2(p_1, l_1)$$

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$$l_3 + p_3 = g_3(p_1, l_1)$$
$$\vdots \tag{1.1}$$
$$l_{m+1} + p_{m+1} = g_{m+1}(p_1, l_1).$$

If $g_k(x, y)$, $k = 2, \ldots, m+1$, are all monomials, the graph is called a *monomial graph*; see [3]. If $g_k(x, y) = x^{k-1}y$, $k = 2, \ldots, m+1$, then the graph is just the original Wenger graph in [2], also denoted by $W_m(q)$. It was shown in [6] that the automorphism group of $W_m(q)$ acts transitively on each of $\mathfrak{P}$ and $\mathfrak{L}$, and on the set of edges of $W_m(q)$. In other words, the graphs $W_m(q)$ are point-, line-, and edge-transitive. It is also shown that, see [7], $W_1(q)$ is vertex-transitive for all q, and that $W_2(q)$ is vertex-transitive for even q. For all $m \geq 3$ and $q \geq 3$, and for $m = 2$ and all odd q, the graphs $W_m(q)$ are not vertex-transitive. Another result of [7] is that $W_m(q)$ is connected when $1 \leq m \leq q-1$, and disconnected when $m \geq q$, in which case it has q^{m-q+1} components, each isomorphic to $W_{q-1}(q)$. In [15], Viglione proved that the diameter of $W_m(q)$ is $2m+2$ when $1 \leq m \leq q-1$. In [2], Cioabă, Lazebnik and Li determined the spectrum of $W_m(q)$.

In this paper we focus on the basic properties of some extensions of Wenger graphs defined as in Eq. (1.1). In Section 2 we first study the spectrum of a general class of graphs such that polynomials $g_k(x, y) \in \mathbb{F}_q[x, y]$ are defined by $g_k(x, y) = f_k(x)y$, and the mapping $\vartheta : \mathbb{F}_q \to \mathbb{F}_q^{m+1}; u \mapsto (1, f_2(u), \ldots, f_{m+1}(u))$ is injective. The eigenvalues of such a graph are determined, however, their multiplicities are reduced to counting certain polynomials with a given number of roots over finite fields. The latter problem is an interesting number theoretical problem, which is expected to be difficult in general. A complete solution in interesting special cases is already significant. In particular, we introduce a new class of bipartite graphs called linearized Wenger graphs. These graphs are denoted by $L_m(q)$, which are defined by Eq. (1.1) together with $g_k(x, y) = x^{p^{k-2}}y$, $k = 2, \ldots, m+1$, and so $f_k(x) = x^{p^{k-2}}$. Using results on linearized polynomials over finite fields, we are able to explicitly determine the spectrum of such graphs when $m \geq e$ in Section 3. Finally we obtain the diameter and girth of linearized Wenger graphs in Section 4 and Section 5, respectively. As a consequence, when $m = e$, this provides a new class of infinitely many connected q-regular expander graphs of q^{2m+2} vertices with optimal diameter $2(m+1)$ when q goes to infinity.

2. The spectrum of general Wenger graphs

In this section we study the basic properties of the class of graphs $\mathfrak{G}$ defined by $g_k(x, y) = f_k(x)y$, where $g_k(x, y)$ is a product of a polynomial in terms of x and the linear polynomial y, for $2 \leq k \leq m+1$. The approach taken in this section follows closely the one in [2].

The following result is proven in much greater generality in [8]. We provide a proof to make the presentation self-contained.

Proposition 2.1. *The graph $\mathfrak{G} = G_q(f_2(x)y, \ldots, f_{m+1}(x)y)$ is q-regular.*

Proof. Given a point P and a line L in V, by definition, $P = (p_1, p_2, \ldots, p_{m+1})$ is adjacent to $L = [l_1, l_2, \ldots, l_{m+1}]$ if and only if the following m equalities hold:

$$\begin{cases} l_2 + p_2 = f_2(p_1)l_1 \\ l_3 + p_3 = f_3(p_1)l_1 \\ \vdots \\ l_{m+1} + p_{m+1} = f_{m+1}(p_1)l_1. \end{cases} \tag{2.1}$$

When the point P is prescribed, (2.1) implies that one can uniquely solve l_k $(k \geq 2)$ from l_1, and thus (2.1) has q solutions. Similarly, when the point L is prescribed, (2.1) implies that one can uniquely solve p_k $(k \geq 2)$ from p_1, and thus (2.1) has q solutions. □

Since $\mathfrak{G}$ is a bipartite graph, its adjacency matrix is of the form:

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

with a matrix N and

$$A^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^TN \end{pmatrix}. \tag{2.2}$$

In order to consider the properties of $\mathfrak{G}$, we define a graph H as follows: the vertex set is $\mathbb{F}_q^{m+1}$ containing all lines in $\mathfrak{G}$, any two lines $L = [l_1, l_2, \ldots, l_{m+1}]$ and $L' = [l'_1, l'_2, \ldots, l'_{m+1}]$ are adjacent if and only if they share a common neighbor point $P = (p_1, p_2, \ldots, p_{m+1})$ in the graph $\mathfrak{G}$ defined above.