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Note

Note on 3-paths in plane graphs of girth 4

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ABSTRACT

An (i, j, k)-path is a path on three vertices u, v and w in this order with $deg(u) \leq i$, $deg(v) \leq j$, and $deg(w) \leq k$. In this paper, we prove that every connected plane graph of girth 4 and minimum degree at least 2 has at least one of the following: a $(2, \infty, 2)$-path, a $(2, 7, 3)$-path, a $(3, 5, 3)$-path, a $(4, 2, 5)$-path, or a $(4, 3, 4)$-path. Moreover, no parameter of this description can be improved. Our result supplements recent results concerning the existence of specific 3-paths in plane graphs.

1. Introduction

In this note we investigate connected plane graphs in which loops and multiple edges are not allowed. We use a standard graph theory terminology according to the book [2]. However we recall some more frequent notions.

Let G be a connected plane graph. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, and $\delta(G)$ (or simply V, E, F, Δ, δ) to denote the vertex set, the edge set, the face set, the maximum degree, and the minimum degree of G, respectively. Faces of G are open 2-cells. The boundary of a face α is the boundary in the usual topological sense. It is a collection of all edges and vertices lying in the closure of a face α that can be organized into a closed walk in the graph G by traversing a simple closed curve just inside the face α. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face α (see [12], p. 101).

The *degree* of a vertex v or a face α, that is the number of edges incident with v or the length of the boundary walk of α, is denoted by $\deg(v)$ or $\deg(\alpha)$, respectively. A *k-vertex* (resp., *k^+-vertex, k^--vertex*) is a vertex of degree k (resp., at least k, at most k). Similarly, a *k-face* (resp., *k^+-face, k^--face*) is a face of degree k (resp., at least k, at most k). An edge uv is of the *type* (i, j) or an *(i, j)-edge*, if $\deg(u) \leq i$ and $\deg(v) \leq j$. A path on three vertices u, v, and w in this order is a *path of type* (i, j, k) or an *(i, j, k)-path* if $\deg(u) \leq i$, $\deg(v) \leq j$, and $\deg(w) \leq k$. Let $w_k(G) = w_k$ be the minimum sum of degrees of vertices of a path on k vertices. The *girth* $g(G) = g$ of G is the length of a shortest cycle in G.

A *normal plane map* is a plane graph in which loops and multiple edges are allowed, but degree of each vertex and each face is at least three.

Already in 1904, Wernicke [20] proved that every normal plane map G of minimum $\delta(G) = 5$ contains a 5-vertex adjacent to a 6^--vertex, and Franklin [11] strengthened this to the existence of a $(6, 5, 6)$-path in such normal plane maps.

It follows from Lebesgue's result in [19] that each normal plane map contains a $(3, 11)$-, or $(4, 7)$-, or $(5, 6)$-edge, where bounds 7 and 6 are sharp. For 3-connected plane graphs, Kotzig [18] proved a precise result: $w_2 \leq 13$. In 1972, Erdös (see [13]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all planar graphs with $\delta \geq 3$. Barnette (see [13]) announced to have proved this conjecture, but first published proof of it is due to Borodin [3].

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Theorem 1 (*Borodin [3]*). *Every normal plane map contains an edge of one of the following types:* (3, 10), (4, 7), *or* (5, 6). *The bounds* 10, 7, *and* 6 *are tight.*

The graphs $K_{1,r}$ and $K_{2,r}$, $r \geq 2$, have only $(1, r)$-edges and $(2, r)$-edges, respectively. They are examples showing that the requirement on minimum degree 3 cannot be omitted when trying to extend Theorem 1 for wider families of plane graphs with minimum degree at least two. Situation changes significantly if we consider plane graphs with girth at least 5. It holds that every connected plane graph G of minimum degree $\delta(G) \geq 2$ contains an edge of type $(2, 5)$ or $(3, 3)$-edge if $g(G) \geq 5$, a $(2, 5)$-edge if $g(G) \geq 6$, a $(2, 3)$-edge if $g(G) \geq 7$ and a $(2, 2)$-edge if $g(G) \geq 11$ (see [16]).

Next step, a study of the existence of 3-paths in plane graphs, has been started in the papers [1,15], and [8]. There are proved the following theorems:

Theorem 2 (*Ando, Iwasaki, Kaneko [1]*). *Every* 3*-connected plane graph satisfies* $w_3 \leq 21$, *which is tight.*

Theorem 3 (*Jendrol' [15]*). *Every* 3*-connected plane graph has a* 3*-path of one of the following: types:* (10, 3, 10), (7, 4, 7), (6, 5, 6), (3, 4, 15), (3, 6, 11), (3, 8, 5), (3, 10, 3), (4, 4, 11), (4, 5, 7), *or* (4, 7, 5).

Theorem 4 (*Borodin et al. [8]*). *Every normal plane map without two adjacent* 3*-vertices lying in two common* 3*-faces has a* 3*-path of one of the following types:* (3, 4, 11), (3, 7, 5), (3, 10, 4), (3, 15, 3), (4, 4, 9), (6, 4, 8), (7, 4, 7), *or* (6, 5, 6). *Moreover, no parameter of this description can be improved.*

Recently, in the paper [16], the study of the existence of 3-paths of a restricted structure in plane graphs with minimum degree at least 2 and a given girth has been started. Namely, there is proved

Theorem 5 (*Jendrol', Maceková [16]*). *Every connected plane graph G of minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq g$ has a* 3*-path of one of the following types:*

(i) $(2, \infty, 2)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, *or* $(3, 3, 3)$, *if* $g = 5$,
(ii) $(2, 2, \infty)$, $(2, 3, 5)$, $(2, 4, 3)$, *or* $(2, 5, 2)$, *if* $g = 6$,
(iii) $(2, 2, 6)$, $(2, 3, 3)$, *or* $(2, 4, 2)$, *if* $g = 7$,
(iv) $(2, 2, 5)$ *or* $(2, 3, 3)$, *if* $g \in \{8, 9\}$,
(v) $(2, 2, 3)$ *or* $(2, 3, 2)$, *if* $g \geq 10$, *and*
(vi) $(2, 2, 2)$, *if* $g \geq 16$.

Some other results related to Kotzig's theorem can be found in the already mentioned papers, in a recent survey [17], and also in the papers [5–7,9], and [14]. Some extensions of Kotzig's Theorem and their application to colouring which was an important stimulating factor for extending Kotzig's Theorem, are discussed in [4].

In this note we supplement the above results on the existence of specific 3-paths in connected plane graphs. Namely, we prove the following:

Theorem 6. *Every connected plane graph G of minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq 4$ contains a* 3*-path of one of the following types:* $(2, \infty, 2)$, $(2, 7, 3)$, $(3, 5, 3)$, $(4, 2, 5)$ *or* $(4, 3, 4)$. *Moreover, no parameter of this description can be improved.*

The proof of Theorem 6 is given in Section 2.

2. Proof of Theorem 6

Proof. Suppose it is not true. Then there exists a counterexample $G = (V, E, F)$ having no 3-path of one of the mentioned types. We continue by the Discharging method (see [10]). Using the consequence of the Euler polyhedral formula

$$\sum_{u \in V(G)} (deg(u) - 4) + \sum_{\alpha \in F(G)} (deg(\alpha) - 4) = -8$$

we put the initial charge $ch(u) = deg(u) - 4$ and $ch(\alpha) = deg(\alpha) - 4$ to the vertices u and faces α of G, respectively. Note that only vertices with degrees 2 and 3 have negative initial charges. These initial charges are locally redistributed according to the following rules:

R1. Every m-face, $m \geq 5$, gives $\frac{1}{3}$ to each incident 3^--vertex.
R2. Every k-vertex, $k \geq 5$, gives 1 to the adjacent 2-vertex v.
R3. Let uvz be a 3-path on the boundary walk of a face α. If $deg(u) \geq 6$, $deg(v) = 2$ and $deg(z) \leq 4$, then u gives additional $\frac{2}{3}$ to v.
R4. Every k-vertex, $k \geq 5$, gives $\frac{1}{3}$ to the adjacent 3-vertex.
R5. Let uvz be a 3-path on the boundary walk of a face α. If $deg(u) \geq 4$, $deg(v) \geq 5$ and $deg(z) \leq 3$, then v gives additional $\frac{1}{6}$ to z, except for the case $deg(v) = 5$ and $deg(z) = 2$.